Wave packet sizes in Quantum Mechanical Scatterings

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Kenzo Ishikawa and Osamu Jinnouchi, arXiv:2509.04539[quantum physics]
“ Wave packet sizes in quantum mechanical scatterings: new perspective”

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1　Why wave packets ?

Principles of the quantum mechanics

1. Superposition principle

   Complex waves　Ψ

2. commutation relation

   variables: pq ≠ qp, but (pq)r=p(qr)

3. Schroedinger equation (Tomonaga-Schwinger)

   $i\,[\partial / \partial\ \sigma]\ \Psi = H\,\Psi$

4. Probability principle

Probability principle

$$P(\alpha \to \beta; T) = |\langle \Psi_\beta (T) | \Psi_\alpha (0) \rangle|^2$$

$\langle \Psi_i | \Psi_i \rangle = 1,\ i = \alpha, \beta$ Normalized states

$\Sigma_{\{\beta\}} P(\alpha \to \beta) = 1$ Absolute values

$0 \leqq P(\alpha \to \beta) \leqq 1$

State vectors follow the Schroedinger equation, and make transitions according to the above probability. Because the probability is non-linear in the wave functions, physical processes reveal non-trivial and rich behaviors that are not expected in classical wave mechanics.

Physical processes

(1)Short distance physics (short range correlation.)

Typical sizes --------atoms, nucleus, quarks,----

1/ 10^10 [m], 1/10^15 [m],

Standard (plane waves) methods with $e^{-\varepsilon |t|} H_{int}$.

(2) Long distance physics(long range correlation.)

Typical sizes------solids, liquid, gas,

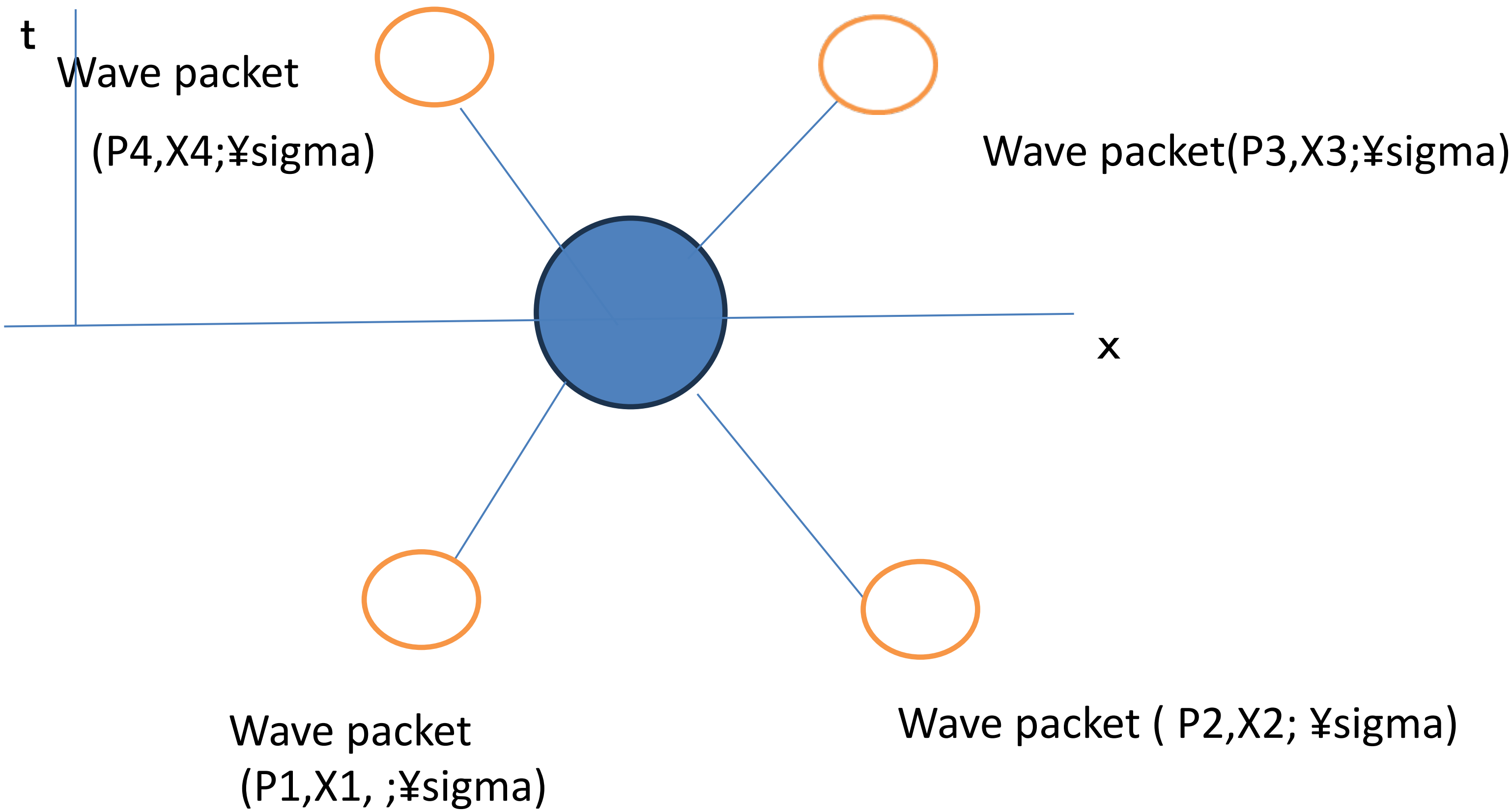
atmosphere, sun,-----

1/10^5[m], 1[m], 10^5[m], -----

New (wave-packet) methods with H_{int}.

Transition of normalized states

- Scattering of wave packets (normalized states)



2　波束　(Gaussian wave packets)

Apply Gaussian wave packets, with which explicit calculations are doable and provide universal properties. (≒ point particle in classical mechanics)

Wave packets are normalized and form complete set.

$$\int dP\, dX \frac{1}{2\pi} |P, X, T_0 > < P.X, T_0 | = 1$$

K.I and T. Shimomura, PTP (2005)

Gaussian wave packets

$$\int dx |\psi_{wave\ packet}(x)|^2 = 1$$

$$\psi_{wave\ packet}(t,x;T_0,X_0) = \int dk N(k) e^{i(k(x-X_0)-E(k)(t-T_0))}$$

$$N(k) = N e^{-\frac{(k-k_0)^2}{2\sigma}}; Gaussian\ wave\ packet$$

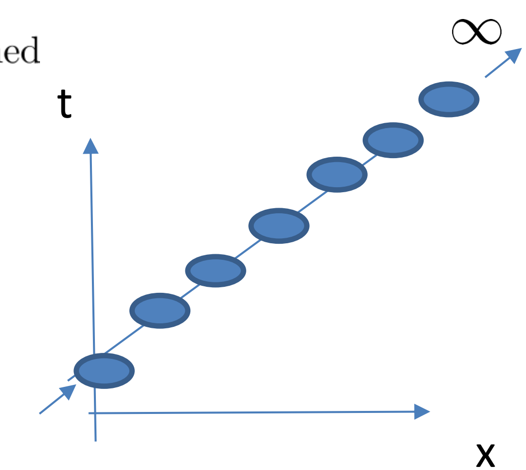
$$\approx N e^{-\frac{(x-X_0-v_0(t-T_0))^2}{2\sigma}+i(k_0(x-X_0)-E(k_0)(t-T_0))}$$

Wave packet size
(唯一のパラメータ)

1. $D(\omega,T) \to G(\omega,T,T_0) = \int_0^T e^{i\omega t - \frac{1}{2\sigma_t}(t-T_0)^2}; |G(\omega,T,T_0)|^2$ is well-defined

2.|<P_2,X_2,T_2: σ_2 |P_1,X_1,T_1; σ_1 >|
≦ (2 ¥sqrt(σ_1 σ_2)/(σ_1+ σ_2))^3

確率＝1 がσ_1＝ σ_2で実現。



K.I and K.Oda, PTEP (2018)

Evolution of wave packets

$Solve\ Schroedinger\ equation\ in\ interaction\ picture,$

$$i\hbar\frac{\partial}{\partial t}|\Psi(t)\rangle_i = gH_{int}(t)|\Psi(t)\rangle_i$$
$$|\Psi(t)\rangle_s = e^{\frac{H_0}{i\hbar}t}|\Psi(t)\rangle_i, H_{int}(t) = e^{-\frac{H_0 t}{i\hbar}}H_{int}e^{\frac{H_0}{i\hbar}t}$$

$$|\Psi(t)\rangle_i = [1 + g\int_0^t \frac{dt'}{i\hbar}H_{int}(t') + g^2\int_0^t \frac{dt'}{i\hbar}\int_0^{t'}\frac{dt''}{i\hbar}H_{int}(t')H_{int}(t'') + \cdots]|\Psi(0)\rangle_i$$

Transition amplitude

Construct the initial and final states with wave packets, and compute the amplitude with Dyson formula.
KI and Y. Tobita, (2013,2014), KI, T.Tajima, and Y.Tobita (2015).
KI and K. Oda(2018), KI,K.Nishiwaki,K.Oda('20,'23) ,KI,Jinnouci,Nishiwaki,Oda(23)

[2-2]　波束の振幅

$\tilde{T}: intersection\ time$

$T = T_{out} - T_{in}$

$$S(T) = \kappa(\prod_A (\pi\sigma_A)^{-3/4} \frac{1}{\sqrt{2E_A}} e^{-\frac{\sigma_t}{2}(\delta\omega)^2 - \frac{\sigma_s}{2}(\delta\vec{p})^2 - R(X)/2} (2\pi\sigma_s)^{3/2} \sqrt{2\pi\sigma_t} G(\tilde{T})$$

$$G(\tilde{T}) = \int_{T_{in}}^{T_{out}} \frac{dt}{\sqrt{2\pi\sigma_t}} e^{-\frac{1}{2\sigma_t}(t-\tilde{T}-i\sigma_t\delta\omega)^2} = G_{bulk}(\tilde{T}) + G_{boundary}(\tilde{T})$$

$$e^{-\frac{\sigma_t}{2}(\delta\omega)^2} G_{bulk} = e^{-\frac{\sigma_t}{2}(\delta\omega)^2} \theta(T_{in}, T_{out}, \tilde{T})$$ ----(A)

$$e^{-\frac{\sigma_t}{2}(\delta\omega)^2} G_{boundary} = \sqrt{\frac{2\sigma_t}{\pi}} \frac{e^{-\frac{(\tilde{T}-T_{in})^2}{2\sigma_t}}}{i\sigma_t\delta\omega + \tilde{T} - T_{in}}$$ ---(B)

$$\delta\omega = \Delta E - \vec{v}_0 \cdot \vec{\Delta} P \qquad \vec{v}_0 = \sigma_s \sum_i \frac{\vec{v}_i}{\sigma_i}, \frac{1}{\sigma_s} = \sum_i \frac{1}{\sigma_i}\ \vec{v}_0 = velocity\ of\ center$$

Wave packet probability　(R(x): positions)

$$P(T) = \kappa^2(\prod_A (\pi\sigma_A)^{-3/2} \frac{1}{2E_A} e^{-\sigma_s(\delta\vec{p})^2 - R(X)} (2\pi\sigma_s)^3 2\pi\sigma_t |G(\tilde{T})|^2$$

$$|G(\tilde{T})|^2 = e^{-\sigma_t(\delta\omega)^2} \theta(T_{in}, T_{out}, \tilde{T}) + \frac{2\sigma_t}{\pi} e^{-\frac{(\tilde{T}-T_{in})^2}{\sigma_t}} \frac{1}{\sigma_t^2(\delta\omega)^2 + (\tilde{T}-T_{in})^2}$$

$1st\ class\ quantiy$　　　　$2nd\ class\ quantiy$

３ 波束の大きさ(One particle states)

(1) Bound states (discrete spectrum)

$H|\psi(E_1)\rangle = E_1|\psi(E_1)\rangle$

$\langle\psi(E_1)|\psi(E_2)\rangle = \delta(E1,E2)$

(2) Stationary states of continuum spectrum

$H|\psi(E_1)\rangle = E_1|\psi(E_1)\rangle$

$\langle\psi(E_1)|\psi(E_2)\rangle = \delta(E1-E2) + \varepsilon$; non-orthogonal

(Landau-Lifshitz, ishikawa-Nishio)

$[\langle\psi(E_1)|(H|\psi(E_2)\rangle)] \neq [(\langle\psi(E_1)|H)|\psi(E_2)\rangle]$

$(AB)C \neq A(BC)$ “ Breaking of associativity” ,KI (2024), KI and Y. Nishio(2025),

Orthogonality of stationary continuum states

(1) Orthogonal :

free system, uniform force (Airy functions),

uniform magnetic filed (Landau levels),

Periodic potential (Bloch’s theorem)

Coulomb potential

(2) non-orthogonal :

short range potentials

Wave packets associated with continuum states

(1) Superposition of free waves and various exceptional potentials.
OK! Use interaction picture with H_{int}.

(2) Superposition of stationary states in short range potentials.
This oscillates with time, and does not represent isolate state. Despite these satisfy the Schroedinger equation rigorously and may look better than (1) mathematically, these do not represent isolate states. Accordingly, (2) is not good from physical reasons.

Note: Kato's proof for the existence of S-matrix was made with (1).

Wave packet sizes (σ)

1. Bound states

   Size of bound state wave function

2. Continuum states

   quantum state loses coherence by interacting with other particles in the enviorements, and has finite coherence length. The coherence length represents the spatial size of the quantum wave function.

   $$\sqrt{\sigma} = l_{MFP}$$

4 a Initial states of scattering

Beam:

Particle in matter is extracted by electric field.

While that is inside of matter, that interacts with many particles. The mean free path (L_mfp) represents average length of one quantum state, i.e., the length of wave function.

The size of wave packet is determined by the mean free path.

L_{mfp}

*Proton by Rutherford scattering

$\sigma(R)=4\pi(e^2/4\pi E)^2 \log\lambda$, $E=\frac{1}{2}Mv^2$

$L=1/\rho\sigma(R)$

= 0.1 [m] ------ E=10 MeV

2.5 x 10^-6 [m] ---- 50 keV

10^-9 [m] ---- 1 keV

*Proton in beam bunch of LHC (proton Coulomb gas)

$L= a_0 \times m_e/m_p =:3 \times 10^{-14}$ [m]

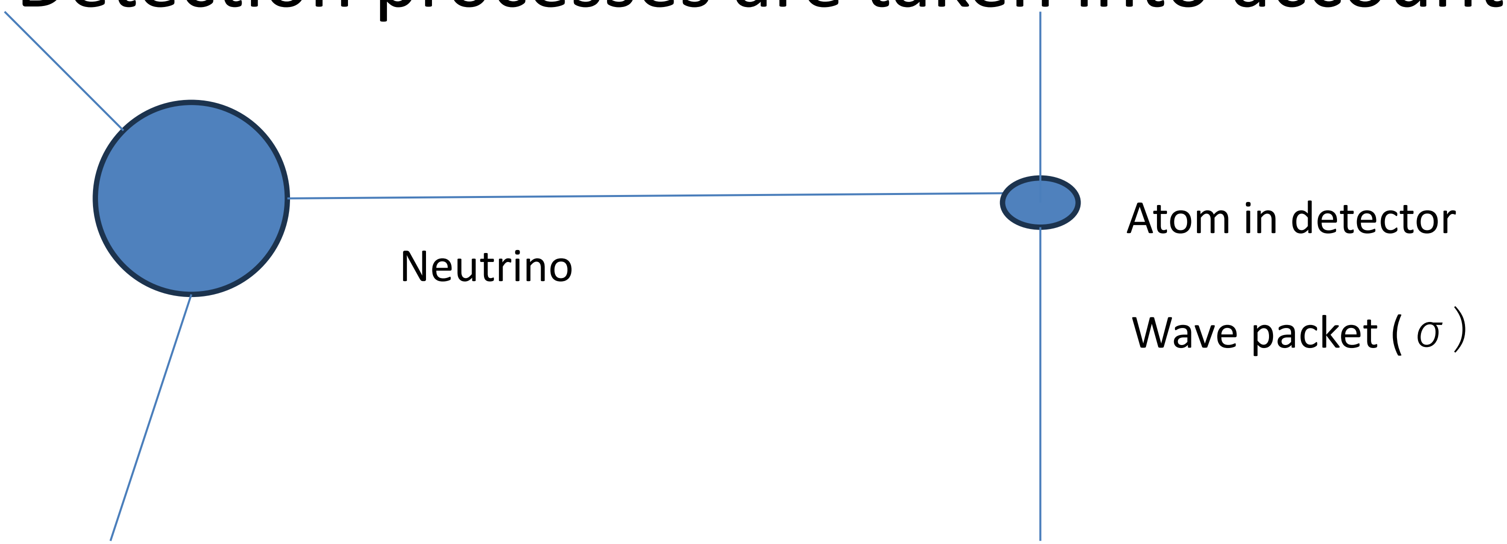
*Neutrino from a pion decay of life time: 10^-8 sec=3x10[m]

*Electron in matter by interaction with imputiries: 10^-7[m]

4 b Final states of scattering

(1) Coherence length(mean free path)

(2) Detection processes are taken into account



$$P(\alpha \to \beta\_0) = 1$$

$$P(\alpha \to \beta\_i) = 0; \text{ for } i \neq 0$$

Overlap with β_0 is 1 and overlap with β_i is 0,

the classical trajectory.

The classical trajectory is realized with

σ(neutrino)= σ(nucleu_d)

σ(photon)= σ(atom_d)

4c Wave packets in natural processes

- Transitions are governed by the probability, irrespective of measurements.

Matter density in the environment is changed

-> wave packet size is changed

-> transition probability is changed

Example: before the decoupling (T=3000K),

$L_e = 5.7 \times 10^{-3}$ [m]

Application 1

- π -> neutrino +lepton;

neutrino is detected by nucleus

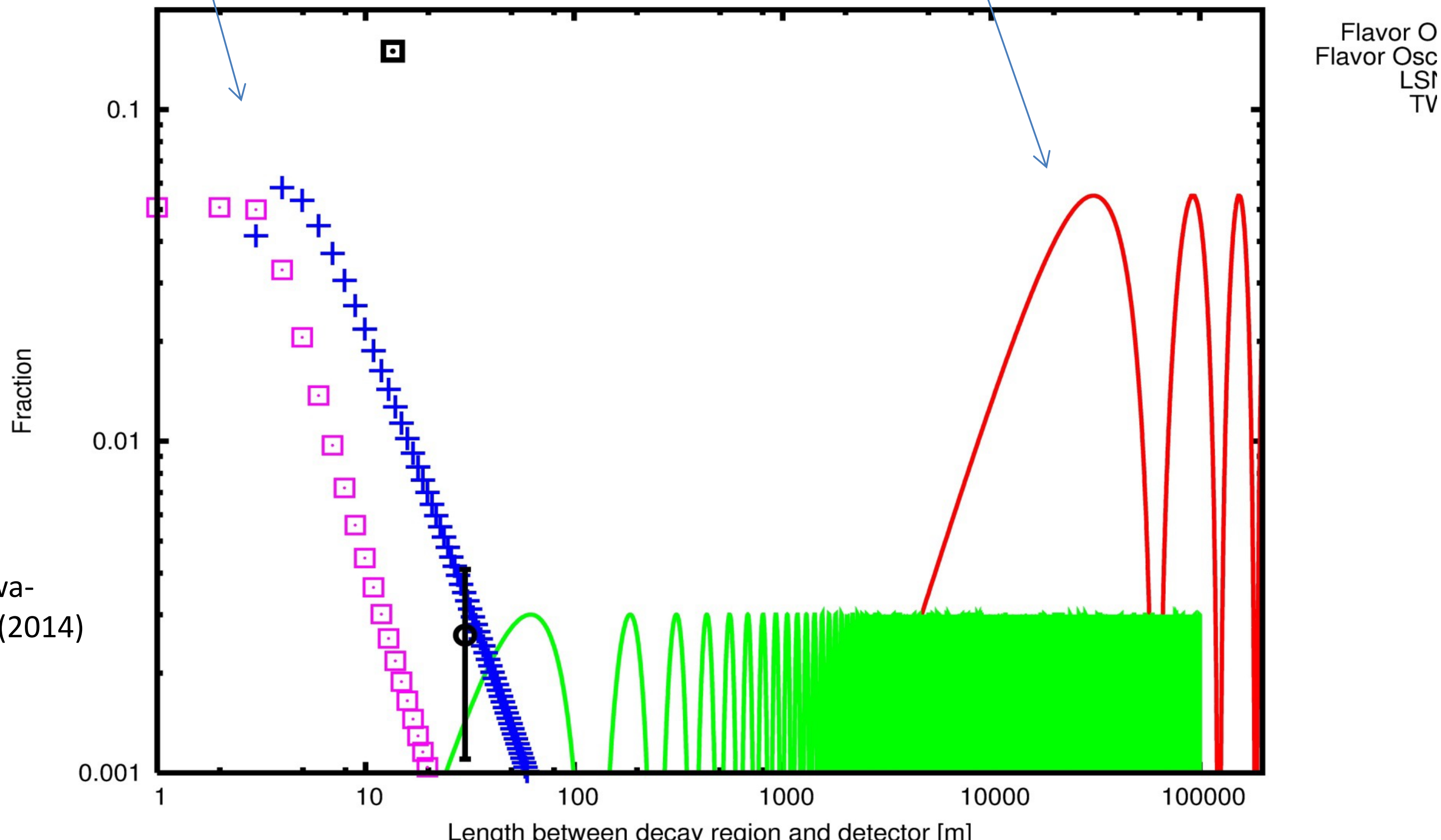
σ (neutrino)= nucleus size

Figure:

Nu_e in pion decay, pi → e+¥nu_e

$P^{(d)}/T$

Flavour oscillation, Γ_0



Ishikawa-tobita,(2014)

Application 2

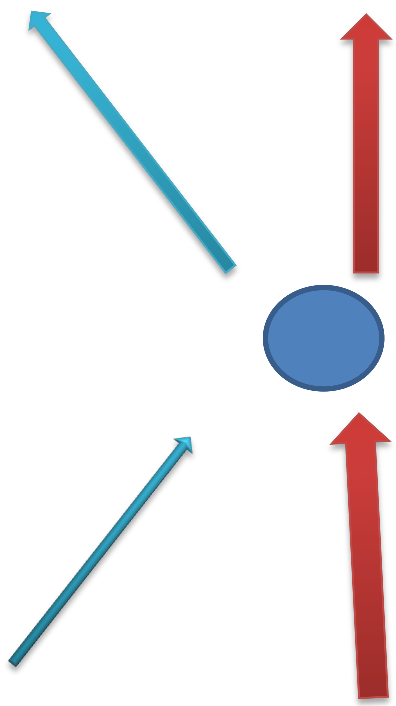
Rayleigh scattering of visible light by molecule

Solar light　太陽表面で生成

波束　 ---- １００ kM

(3-1) Rayleigh scattering

light ＋molecule —> light ＋molecule
（k_1,X_1) +M(0, X_M) ----> (k_2,X_2) + +M(0,X_M)



|0, X_M,T_2)

|K_2, X_2,T_2)

σ (light
)

State

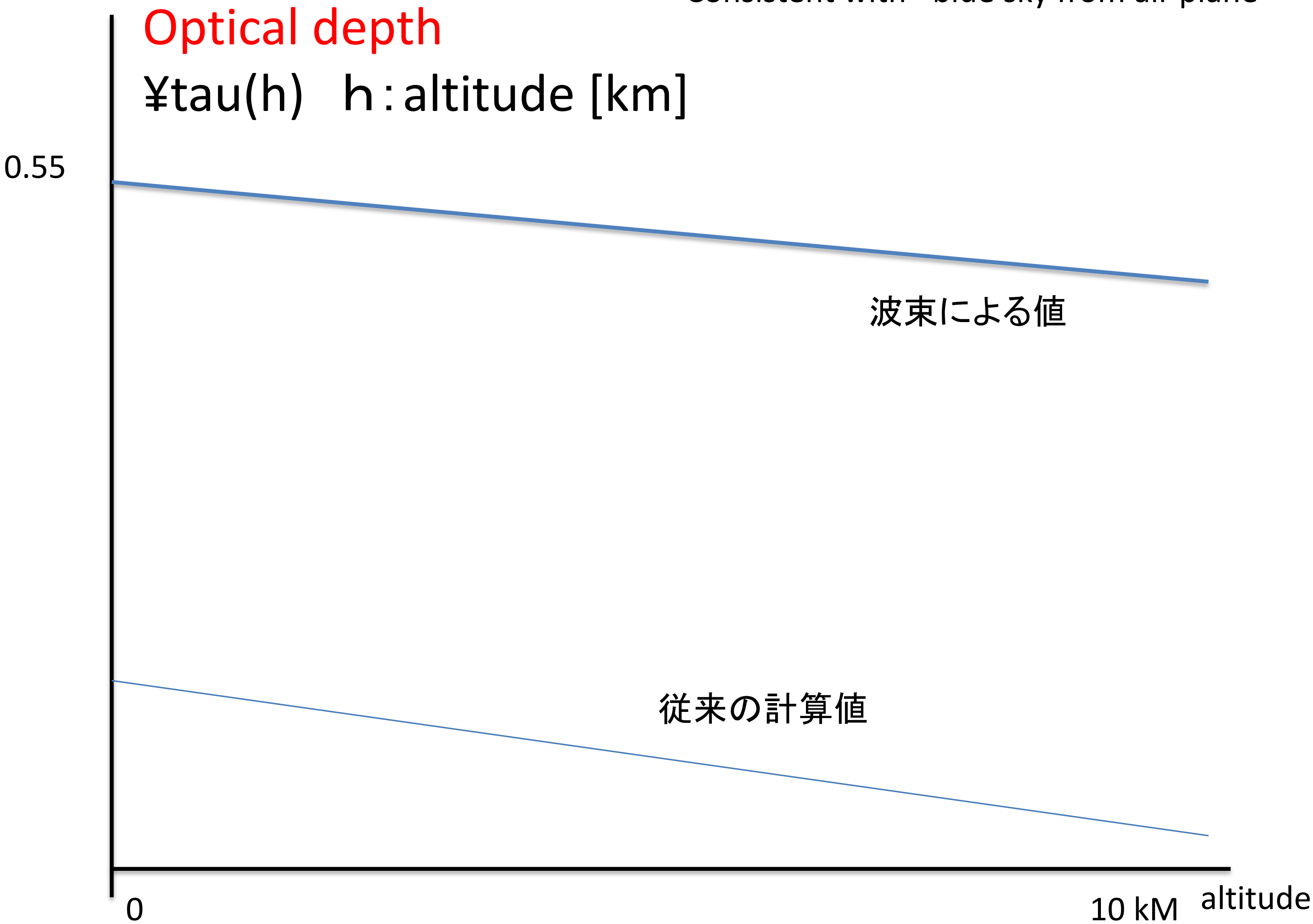
Momentum（P) and
position（X)

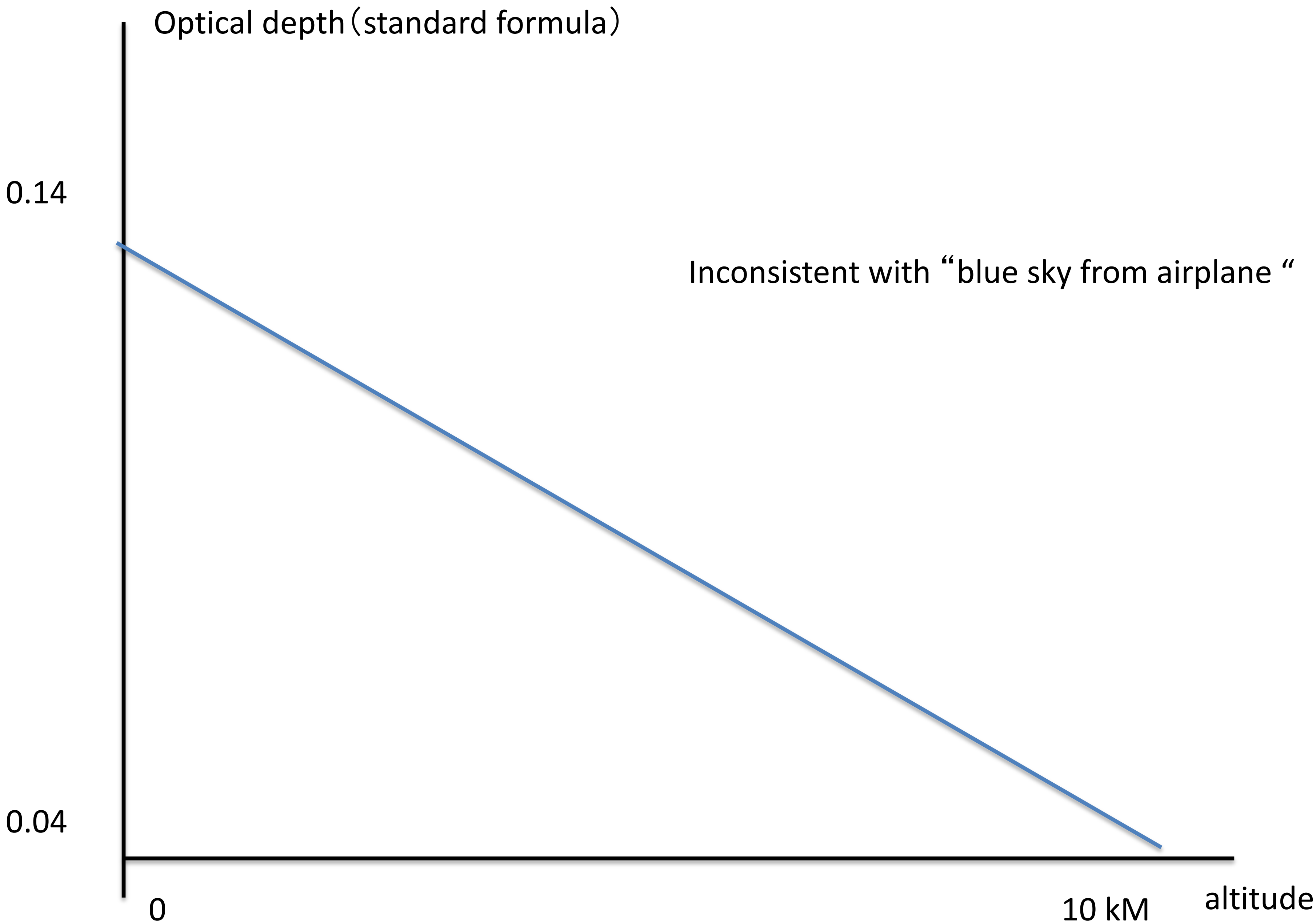
|0, X_M,T_1)

|K_1, X_1,T_1)

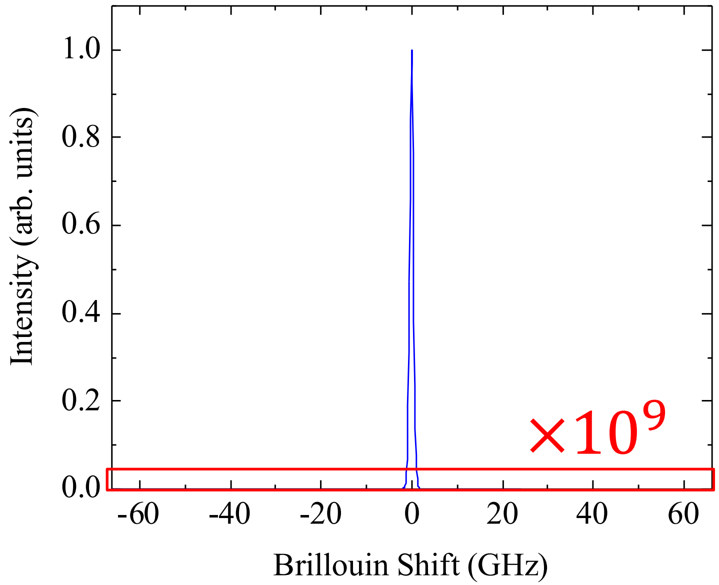
σ (light)

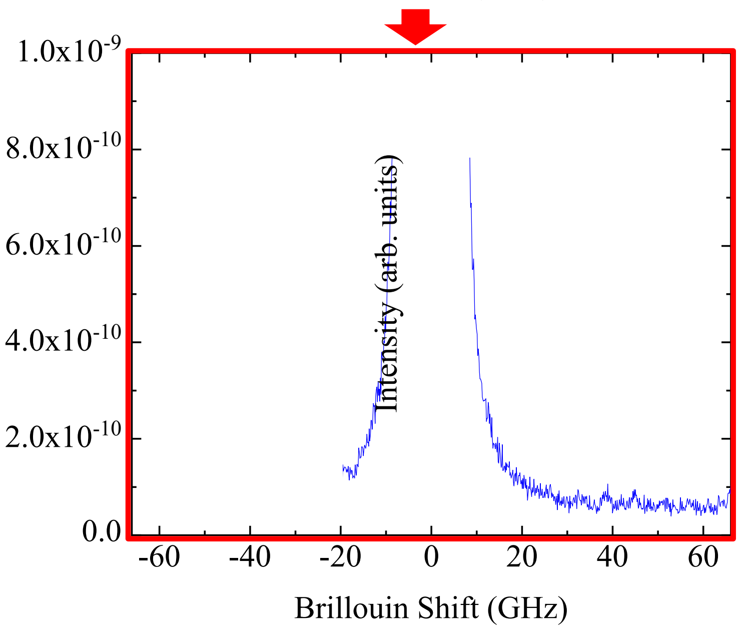
Consistent with “blue sky from air plane ”




Optical depth（standard formula）
0.14
Inconsistent with “blue sky from airplane “
0.04
0
10 kM
altitude

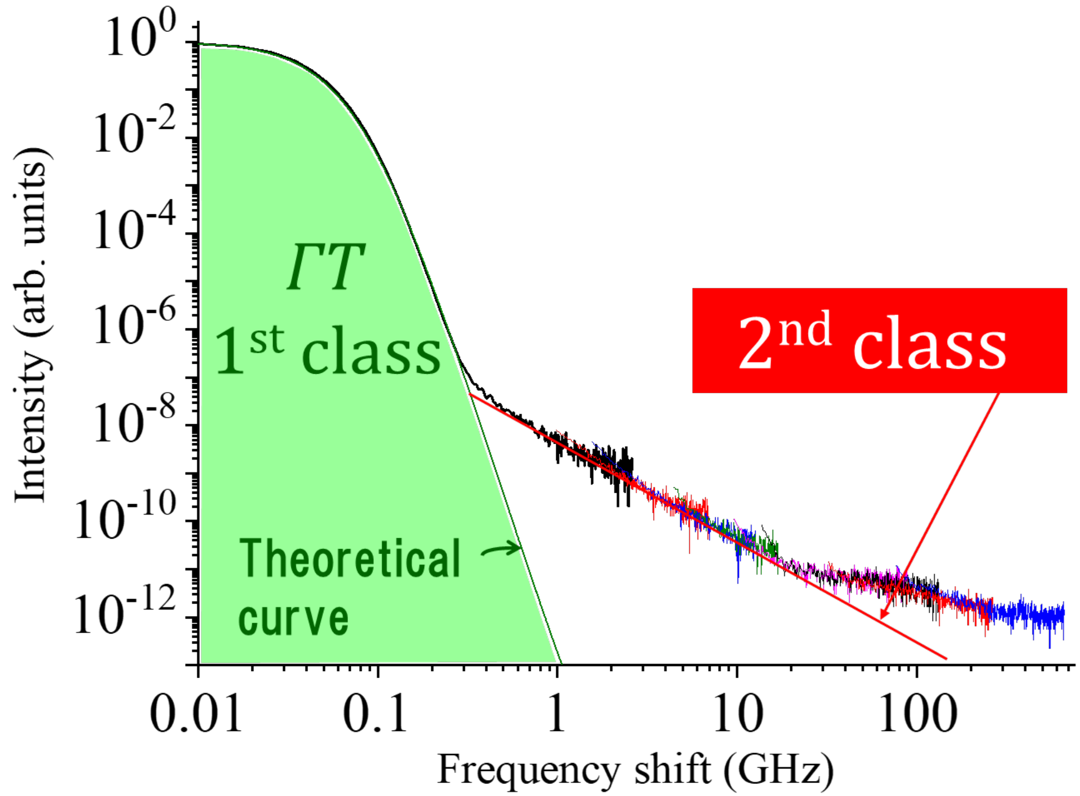
Rayleigh scattering spectrum （takesada)





$$P(T) = \left|\langle \psi_{\text{out},\beta,T} | \psi_{\text{in},\alpha,0} \rangle\right|^2$$
$$= \Gamma T + P^{(d)}$$

1st 2nd



Application 3

- Positron annihilations

On-going experiments. (Jinnouchi)

・K.Ishikawa ,O.Jinnouchi,et al, “ On experimental confirmation of the corrections to Fermi’s golden rule” ,Prog.Theor.Exp.Phys.033B02,(2019)
doi: 10.1093/ptep/ptz006
・R.Ushioda, et., al,” Search for the correction term to the Fermi’s golden rule in positron annihilation”, Prog.Theor. Exp.Phys.043C01
doi:10.1093/ptep/ptaa018(2020)

References:

場の理論（１）

・K.Ishikawa and T.Shimomura, “ Generalized S-matrix in mixed representation”, Prog.Theor. Phys.114,1201 (2006)

・K.Ishikawa and Y.Tobita,“ Finite-size correction to Fermi’s golden rule: I.Decay rates”, Prog.Theor. Exp.Phys,073B02, doi:10.1093/ptep/ptt049 (2013)

・K.Ishikawa and Y.Tobita, “Matter-enhanced transition probabilities”, Ann of Phys. 344,118(2014). doi:10.1016/j.aop.2014.02.007 (2014)

・K.Ishikawa , T.Tajima, Y.Tobita, “Anomalous radiative transitions”, Prog.Theor.Exp.Physics.013B02, doi:10.1093/ptep/ptu168 (2015)

場の理論（２）

・K.Ishikawa and K.-Y.Oda, “ Particle decay in Gaussian wave-packet formalism revised”, Prog.Theor.Exp.Phys.123B01, doi:10.1093/ptep/pty127 (2018)

・K.I, K.Nishiwaki, K-Y.Oda, “ Scalar scattering amplitude in the Gaussian wave-packet formalism” Prog.Theor.Exp.Phys.103B04,(2020) doi:org/10.1093/ptep/ptaa127

・K.I, K.Nishiwaki, K-Y.Oda, “ Scalar scattering amplitude in the Gaussian wave-packet formalism” Prog.Theor.Exp.Phys.103B04,(2020)
doi:org/10.1093/ptep/ptaa127
・K.I, O.Jinnouchi,K.Nishiwaki, K.Oda, Eur.Phys.C(2023)83:978
doi:https://doi.org/10.1140/epjc/s10052-023-12077-7
・K.Ishikawa ,K.Nishiwaki, K.Oda, “New effect in wave-packet scattering of quantum fields”, Phys. Rev. D 108,096013(2023)
https://doi.org/10.1103/PhysRevD.108.096013

ポテンシャル散乱

・K.Ishikawa, “Potential scatterings in the L^2 space: (2) Rigorous scattering probability of wave packets” Annals of Phys. 460,(2024), 169571,
https://doi.org/10.1016/j.aop.2023.169571
・K.Ishikawa and Y.Nishio, “Overlap integrals of continuum stationary states” Annals of Phys. 469,(2024), 169750,
https://doi.org/10.1016/j.aop.2024.169750

Born の確率解釈との関連

レクチャー量子力学 I , II, 石川健三、裳華房 (東京) 2020

光合成

・N.Maeda et al, “Finite-size corrections to the excitation energy transfar in a massless scalar interaction model” , Prog.Theor.Exp.Phys.053J01, doi:10.1093/ptx/ptt066 (2017)

物性

・ K.Ishikawa and Y.Tobita, “Topological interaction of neutrino with photon in a magnetic field-Elactroweak Hall effect ”,Physics Open, 17(2023) 100174, https://doi.org/10.1016/j.physo.2023.100174

・K.Ishikawa,” Magnetization without spin: Effective Lagrangian of itinerant electrons” Nucl.Physics B1007,(2024), 11663,https://doi.org/10.1016/j.nuclphysb.2024.116663

・K.Ishikawa and M.Takesada,”New class of quantum transitions exhibiting large-scale intercorrelations” in preparation.

まとめ

波束の大きさは、環境に依存した値となる。
そのため、遷移現象の状況を正確に把握し、始・終状態の波束の大きさを求める。
この際、環境や測定器に関する解析が一般に必要である。

これらに基づいて得られた波束と、相互作用を使い、得られる遷移確率の絶対値が、自然現象や測定値を決定する。

T_{int} and R are

$$T_{int} = -2\frac{\sigma_t}{\sigma}[(\vec{v}_{\gamma_2}(\vec{X}_{\gamma_2} - \vec{v}_{\gamma_2}T_1) + \vec{v}_{\gamma_1}(\vec{X}_{\gamma_1} - \vec{v}_{\gamma_1}T_0))$$
$$-\frac{1}{4}(\vec{v}_{\gamma_1} + \vec{v}_{\gamma_2})[\vec{X}_M + \frac{1}{2}(\vec{X}_{\gamma_1} - \vec{v}_{\gamma_1}T_0) + \frac{1}{2}(\vec{X}_{\gamma_2} - \vec{v}_{\gamma_2}T_1)]], \tag{H49}$$

and

$$\frac{R}{2} = \frac{1}{\sigma}\vec{X}_M^2 + \frac{1}{2}[\frac{1}{\sigma}(\vec{X}_{\gamma_2} - \vec{v}_{\gamma_2}T_1)^2 + \frac{1}{\sigma}(\vec{X}_{\gamma_1} - \vec{v}_{\gamma_1}T_0)^2] - \frac{\sigma_s}{2}(\delta\vec{p})^2$$
$$-\frac{1}{2\sigma}[\vec{X}_M + \frac{1}{2}(\vec{X}_{\gamma_2} - \vec{v}_{\gamma_2}T_1) + \frac{1}{2}(\vec{X}_{\gamma_1} - \vec{v}_{\gamma_1}T_0) + i\frac{\sigma}{2}(\vec{p}_{\gamma_1} - \vec{p}_{\gamma_2})]^2 \tag{H50}$$
$$-\frac{2\sigma_t}{\sigma^2}[[\vec{v}_{\gamma_2}(\vec{X}_{\gamma_2} - \vec{v}_{\gamma_2}T_1) + \vec{v}_{\gamma_1}(\vec{X}_{\gamma_1} - \vec{v}_{\gamma_1}T_0)]$$
$$-\frac{1}{4}(\vec{v}_{\gamma_1} + \vec{v}_{\gamma_2})[\vec{X}_M + \frac{1}{2}(\vec{X}_{\gamma_2} - \vec{v}_{\gamma_2}T_1) + \frac{1}{2}(\vec{X}_{\gamma_1} - \vec{v}_{\gamma_1}T_0) + i\frac{\sigma}{2}(\vec{p}_{\gamma_1} - \vec{p}_{\gamma_2})]]^2.$$