Universal constraint on relaxation times in Open Quantum Systems

-- Physics of complete positivity and beyond…

2026.1.19. @ Tokyo Woman's Christian University

Gen Kimura (Shibaura Institute of Technology)

Based on (joint) works:
*G. K., Phys. Rev. A 66, 062113 (2002).
*G. K., S. Ajisaka, K. Watanabe, Open Syst. Inform. Dynam. 24(4): 1-8 (2017).
*D. Chruscinski, G. K., A. Kossakowski, Y. Shishido (2020), Phys. Rev. Lett. 127, 050401 (2021).
*D. Chruscinski, R. Fujii, G. K., H. Ohno, Linear Algebra Appl. 630, 293-305 (2021).
*D. Chruscinski, G. K., F. Mukhamedov, J. Phys. A: Math. Theor. 57 185302 (2024).
*P. Muratore-Ginanneschi, G. K., D. Chruściński, J. Phys. A: Math. Theor. 58 045306 (2025).
*D. Chruściński, F. vom Ende, G. K., P. Muratore-Ginanneschi, Rep. Prog. Phys. 88, 097602 (2025).
*F. vom Ende, D. Chruściński , G. K., P. Muratore-Ginanneschi, Linear Algebra Appl., 730. 262 (2026).

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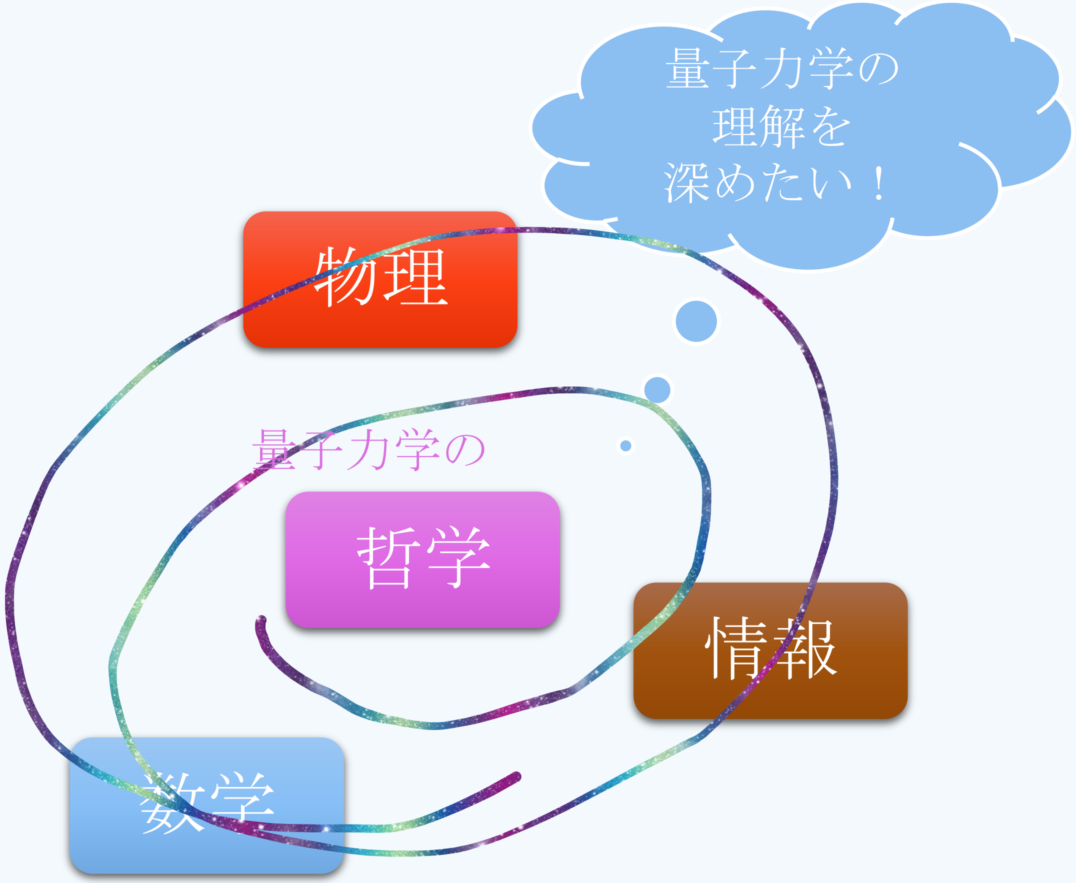
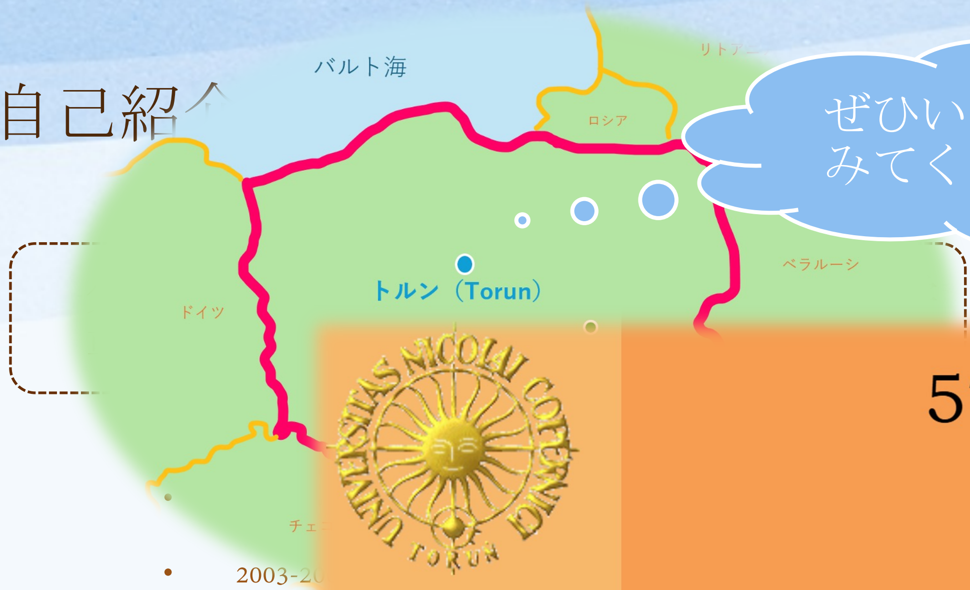
- 2026 - （他大に移動予定…）

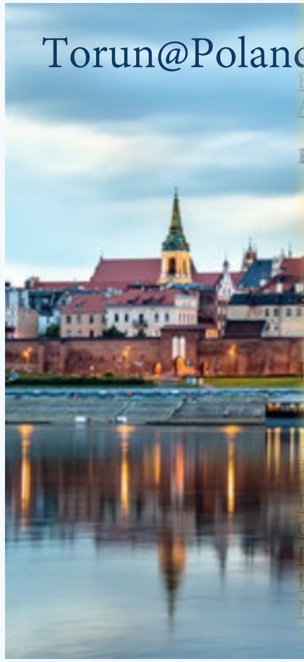


研究分野？

自己紹介



ぜひいつか訪れて
みてくだわさい！



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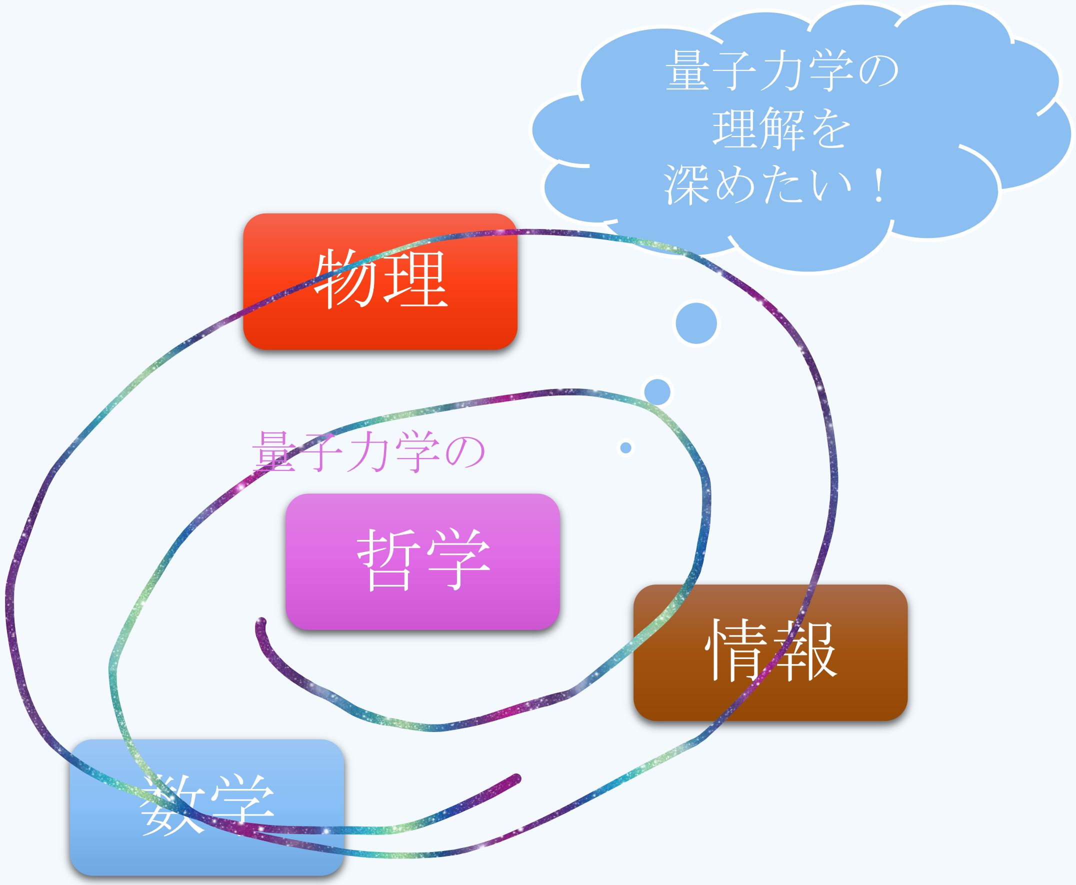

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物理学会誌 2026年1月 (第81巻, 1号)

解　説

ベル定理を詳解する――理論と実験

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- 2026 - （他大に移動予定...）

量子開放系理論

今日の話

不確定性関係

一般確率論

ベル定理

最近の研究テーマ

Quantum Foundations

第三回 2026/3/5,6

@大阪大学・豊中キャンパス南部ホール

TOP MESSAGE INFORMATION APPLICATION FORM PROGRAMS AWARD NEW ORGANIZERS English

＊2024年度から開催された
量子基礎論の定期研究会

＊若手が気軽に発表できる会

＊3回目より学生発表賞創設

＊対面・オンラインの
ハイブリッド研究会

*口頭発表 2/13
*ポスター発表 2/19
*参加登録3/1

Outline

1) Brief introduction to Quantum Dynamical Semigroup

-- CPTP map as a time evolution map
-- Why need CP condition ??
-- Quantum (CP) Dynamical Semigroup and GKLS master equation

Gorini, Kossakowski, Sudareshan (1976)

Lindblad (1976)

2) Goal: To find a test to check the validity of CP condition !!

3) Universal constraints of relaxation rates for CP dynamical semigroup $c(d)\Gamma_\alpha \leq \sum_{\beta=1}^{d^2-1} \Gamma_\beta$

4) Universal constraints of relaxation rates for n-positive dynamical semigroup

5) Conclusions and Discussion

Review of Open Quantum Mechanics necessary for this talk

- Quantum State: a density operator ρ on Hilbert space*:

$$\mathrm{tr}\ \rho = 1,\ \rho \geq 0$$

$$\forall \psi \in \mathcal{H}, \langle \psi | \rho \psi \rangle \geq 0$$

* In this talk, we restrict to a d-level quantum system ($d < \infty$)

System of Interest

Review of Open Quantum Mechanics necessary for this talk

* Quantum State: a density operator ρ on Hilbert space*:

$$\mathrm{tr}\ \rho = 1,\ \rho \geq 0$$

$\forall \psi \in \mathcal{H}, \langle \psi | \rho \psi \rangle \geq 0$

* In this talk, we restrict to a d-level quantum system ($d < \infty$)

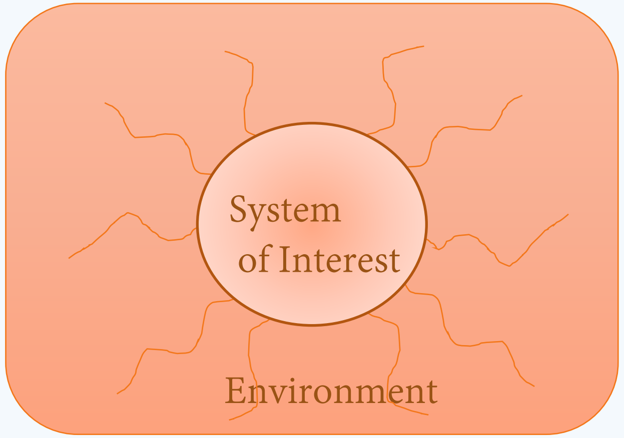
Is CP condition really necessary??

* Tim

Isolated System ⇔ Unitary Evolution $\rho \mapsto \Phi(\rho) = U \rho U^\dagger$

based on von Neumann-Schrödinger equation: $\frac{d}{dt}\rho = -\frac{i}{\hbar}[H, \rho]$

Open System ⇔ Completely Positive Map $\rho \mapsto \Phi(\rho) = \sum_i V_i \rho V_i^\dagger$



* Composite System is described by a Tensor Product Hilbert Space

$$\mathcal{H}_{\mathrm{tot}} = \mathcal{H}_{\mathrm{sys}} \otimes \mathcal{H}_{\mathrm{env}}$$

CPTP map as a Time evolution map

Initial State

$$\rho \mapsto \Phi(\rho)$$

Final State

Hence, Time evolution map is a Positive TP map

Operationally, Φ should

(i) preserve Probabilistic Mixture

$$\Phi(p\rho + (1-p)\sigma) = p\Phi(\rho) + (1-p)\Phi(\sigma)$$

(i) Φ is linear map

(ii) map a density operator to a density operator

$$\mathrm{tr}\rho = 1, \rho \geq 0 \Rightarrow \mathrm{tr}\Phi(\rho) = 1, \Phi(\rho) \geq 0$$

(ii)-1 Φ is trace preserving (TP) map

$$\mathrm{tr}\rho = \mathrm{tr}\Phi(\rho)$$

(ii)-2 Φ is positive (preserving) map

$$\rho \geq 0 \Rightarrow \Phi(\rho) \geq 0$$

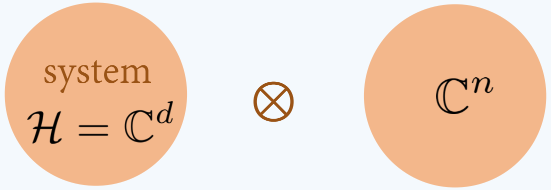
Stronger Condition of Complete Positivity condition

CPTP map as a Time evolution map

Initial State $\rho \mapsto \Phi(\rho)$ Final State

[Def] Φ is n-positive (n =1,2,3,…,) if

$$\Phi \otimes \mathrm{Id}_n \text{ is positive on } \mathcal{H} \otimes \mathbb{C}^n$$

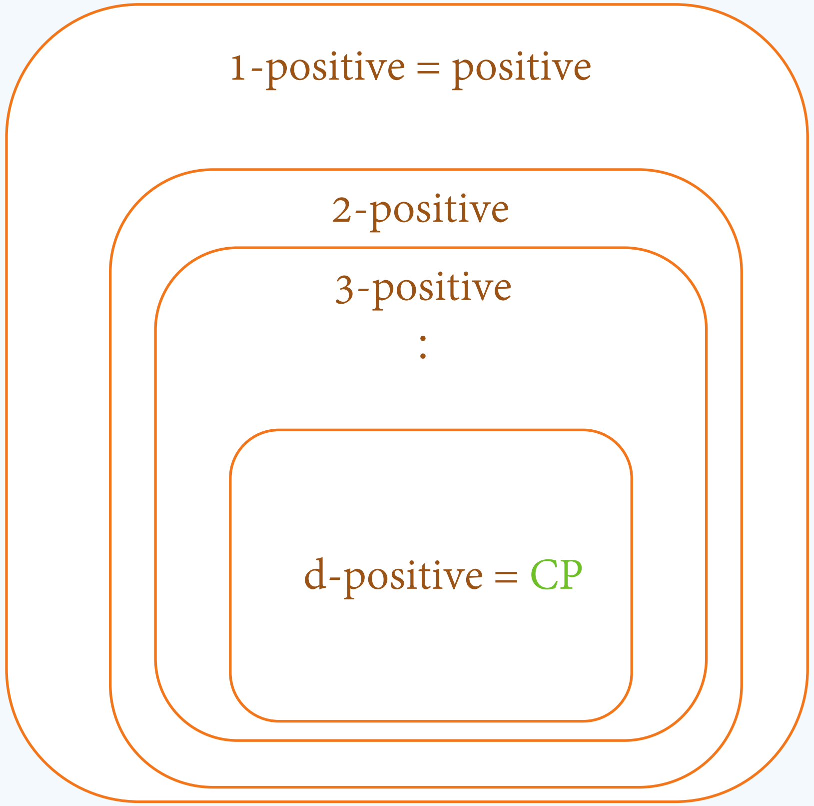


[Def] Φ is **completely positive** if Φ is n-positive **for all n =1,2,3,…**

[Remarks] (i) 1-positive ⇔ positive

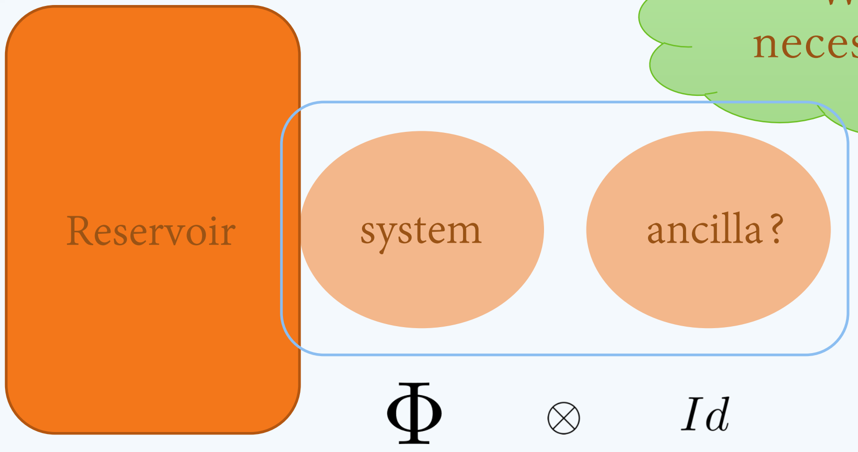
(ii) d-positive ⇔ CP

(iii) for all k =1,…,d-1, there are k-positive but not (k+1)-positive



[Thm] (Choi-Kraus Representation) Φ is completely positive iff $\Phi(X) = \sum_k V_k X V_k^\dagger$

Why we need complete positivity condition?



Why necessary?

drop the time parameter. We further assume that S_2 is a closed system, i.e. its dynamics is given by a Hamiltonian H_2. We put $H_2=0$ for the moment. Then we ask: can the map Φ_1 be extended to a positive map $\Phi: B(\mathscr{H}) \to B(\mathscr{H})$ where $\mathscr{H}=\mathscr{H}_1 \otimes \mathscr{H}_2$, such that S_2 is unaffected? This is obviously so when the dynamics of S_1 is Hamiltonian. Then the dynamics of S_1+S_2 is given by the Hamiltonian $H=H_1 \otimes I_2$. Φ is defined by

$$\Phi(X \otimes Y)=\Phi_1(X) \otimes Y \qquad (2.1)$$

Lindblad (1976)

GKS (1976)

sociated to S and to R, respectively. Assume that $S+R$ has been initially prepared in a product state $\rho\otimes\sigma$, $\rho\in\mathcal{T}(\mathcal{H}_S)$, $\sigma\in\mathcal{T}(\mathcal{H}_R)$, in which S and R are uncorrelated. The Heisenberg reduced dynamics of S, $\Phi: t\to\Phi_t:\mathcal{B}(\mathcal{H}_S)\to\mathcal{B}(\mathcal{H}_S)$, $t\in\mathbb{R}^+$, is defined by

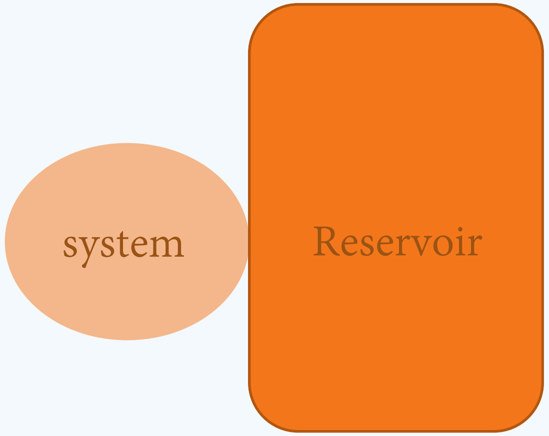
$(\Phi_t A)]$, (1.3)

). It is easy to see
proof can be found

Initial Correlations?

$$\Phi(\rho_S) = \mathrm{Tr}_R U \rho_S \otimes \rho_R U^\dagger$$

Should always be satisfied?



But according to discipline of natural science,

we must rely on **experiments** in order to determine something is right or wrong !!

Experiments

What is the **physics of CP** condition ??

Bell Theorem

Reality

Locality

"Free will"

Bell's Inequality

$$\langle AC\rangle + \langle AD\rangle + \langle BC\rangle - \langle BD\rangle \leq 2$$

Goal

CP

Quantum Dynamical Semigroup

Markovianity

Experimentally testable condition.

$$c(d)\Gamma_\alpha \leq \sum_{\beta=1}^{d^2-1} \Gamma_\beta$$

Direct physical manifestation of CP condition ??

Redundant!

Quantum Process Tomography ⇒ Check CP condition

“Complete Positivity Witness”

Redundant!

Quantum State Tomography ⇒ Check Entanglement Criterion

Entanglement Witness

Quantum dynamical semigroup … General Markovian CP quantum dynamics

1) Completely Positive Trace Preserving Map $\rho \mapsto \rho_t = \Lambda_t \rho$

2) One parameter (time) Dynamical Semigroup $\Lambda_{t+s} = \Lambda_t \Lambda_s$ (∀

Markov property

We should call
GKLS (or GKSL)
master equation!

Hille-Yoshida (1948)

Master equation $\dfrac{d\rho}{dt} = \mathcal{L}\rho$ s.t. $\Lambda_t = \exp(t\mathcal{L})$

[Thm] (GKLS 1976) Generator of quantum dynamical semigroup is always written

$$\mathcal{L} = \mathcal{H} + \mathcal{D}$$

＊Hamiltonian Part

$$\mathcal{H}(\rho) = -i[H, \rho]$$ where $H = H^\dagger$

(effective) Hamiltonian

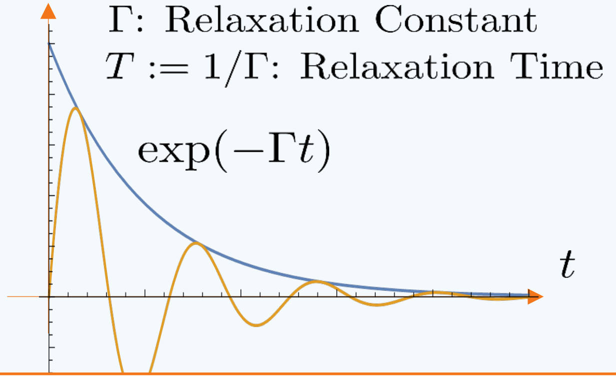
＊Dissipative Part:

$$\mathcal{D}(\rho) = \frac{1}{2}\sum_k (2L_k \rho L_k^\dagger - L_k^\dagger L_k \rho - \rho L_k^\dagger L_k)$$

L_k: Jump/Noise Operator

Physics of Markovian Master equation

We focus on Relaxation Times !



Time Evolution of
any physical quantities

= "super position" of

Trace Preserving Property

There are d^2-1 numbers of decaying time scale !

Solution is dete... Eigenvalues of Generator:

$$\lambda_0 = 0 \quad \& \quad \lambda_\alpha = -\Gamma_\alpha + i\omega_\alpha \quad (\alpha = 1, \ldots, d^2 - 1)$$

Experimentally Accessible quantities !

Positive !

$\mathrm{Re}\lambda_\alpha$ Relaxation Rates/Cons

$T_\alpha := 1/\Gamma_\alpha$ Relaxation Times

Constraints from CP condition

For 2 level (qubit) system

[Example] $$\mathcal{L}\rho = \sum_{k=1}^{3} \gamma_k(\sigma_k \rho \sigma_k - \rho)$$

$$\Gamma_1 = \gamma_2 + \gamma_3,\ \Gamma_2 = \gamma_3 + \gamma_1,\ \Gamma_3 = \gamma_1 + \gamma_2,$$

$\rho = \frac{1}{2}(\mathbb{I} + \boldsymbol{b}(t) \cdot \boldsymbol{\sigma})$ where $\boldsymbol{b}(t) = (e^{-\Gamma_1 t} b_1, e^{-\Gamma_2 t} b_2, e^{-\Gamma_3 t} b_3)\ (i = 1, 2, 3)$

$\Lambda_t = \exp(\mathcal{L}t)$ Positive Trace Preserving $\Leftrightarrow \Gamma_k \geq 0\ (k = 1, 2, 3)$

Trivial constraints ..

There are no constraints for Relaxation Times without CP condition !

Constraints from CP condition

For 2 level (qubit) system

[Example] $$\mathcal{L}\rho = \sum_{k=1}^{3} \gamma_k (\sigma_k \rho \sigma_k - \rho)$$

$$\Gamma_1 = \gamma_2 + \gamma_3,\ \Gamma_2 = \gamma_3 + \gamma_1,\ \Gamma_3 = \gamma_1 + \gamma_2,$$

$$\rho = \frac{1}{2}(\mathbb{I} + \boldsymbol{b}(t) \cdot \boldsymbol{\sigma}) \quad \text{where} \quad \boldsymbol{b}(t) = (e^{-\Gamma_1 t} b_1, e^{-\Gamma_2 t} b_2, e^{-\Gamma_3 t} b_3)\ (i = 1, 2, 3)$$

$\Lambda_t = \exp(\mathcal{L}t)$ Completely Positive Trace Preserving $\Leftrightarrow \gamma_k \geq 0$

$$\Leftrightarrow \Gamma_1 \leq \Gamma_2 + \Gamma_3,\ \Gamma_2 \leq \Gamma_3 + \Gamma_1,\ \Gamma_3 \leq \Gamma_1 + \Gamma_2,$$

Non-trivial constraints !!

Constraints from CP condition

For 2 level (qubit) system

[Theorem] (GKS 1976) (Kimura 2002) For Arbitrary 2-level GKLS,

$$\text{If } [H, D] = 0, \quad \Gamma_1 + \Gamma_2 \geq \Gamma_3,\ \Gamma_2 + \Gamma_3 \geq \Gamma_1,\ \Gamma_3 + \Gamma_1 \geq \Gamma_2$$

Pauli Master Equation

In terms of the γ_i's, (3.4) (a) can be written

$$\gamma_1 + \gamma_2 \geqslant \gamma_3, \quad \gamma_2 + \gamma_3 \geqslant \gamma_1, \quad \gamma_3 + \gamma_1 \geqslant \gamma_2, \tag{3.5}$$

showing that no two relaxation times can be much longer than the third.

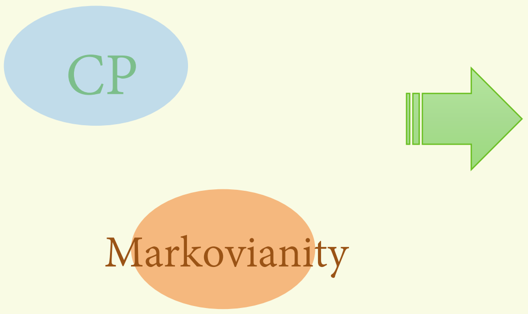
Constraints from CP condition

For 2 level (qubit) system

[Theorem] (GKS 1976) (Kimura 2002) For Arbitrary 2-level GKLS,

$$2\Gamma_\alpha \leq \sum_{\beta=1}^{3} \Gamma_\beta \quad (\forall \alpha = 1,2,3)$$

$$\Gamma_1 + \Gamma_2 \geq \Gamma_3, \ \Gamma_2 + \Gamma_3 \geq \Gamma_1, \ \Gamma_3 + \Gamma_1 \geq \Gamma_2$$



Goal

Experimentally testable condition

$$2\Gamma_\alpha \leq \sum_{\beta=1}^{3} \Gamma_\beta$$

"Witness" of Complete Positivity !!

GENERAL

Constraints from CP condition

For 2 level (qubit) system

> [Theorem] (GKS 1976) (Kimura 2002) For Arbitrary 2-level GKLS,
>
> $$2\Gamma_\alpha \leq \sum_{\beta=1}^{3} \Gamma_\beta \ (\forall \alpha = 1,2,3)$$

Experimentally Famous Relations*

[Case 1] Eigenvalues: $-\Gamma_L, -\Gamma_T \pm i\omega$ $\Rightarrow \Gamma_1 = \Gamma_L, \Gamma_2 = \Gamma_3 = \Gamma_T$

$T_L = 1/\Gamma_L$ Longitudinal Relaxation Time

$T_T = 1/\Gamma_T$ Transverse Relaxation Time

$$\Rightarrow 2T_L \geq T_T$$

[Case 2] Eigenvalues: $-\Gamma_1, -\Gamma_2, -\Gamma_3$ $\leftarrow$ Let us know such experiments!

* Flakowski et al. (2016)

How about general d-level system ??

Constraints from CP condition

For general d level (qudit) system

"Quantum Dynamical Semigroups and Applications"

by Alicki and Lendi (1987)

"Constraints on relaxation rates for d-level quantum systems"

by S. Schirmer, A. Solomon (2004)

etc.

Not real relaxation rates but just parameters in GKLS generator!
Not universal results, dependent on each model!

[Theorem] (Wolf and Cirac 2008): For d-level GKLS without Hamiltonian Part

$$\frac{2}{d}\Gamma_\alpha \leq \sum_{\beta=1}^{d^2-1} \Gamma_\beta \ (\forall\alpha = 1, \ldots, d^2 - 1)$$

GOAL: Find general constraints on Relaxation Times

universally valid for any d-level (qdit) GKLS master equation !

CP requires that
No single relaxation rate
cannot be too large ! !

d=2 [Thm*] $$2\Gamma_\alpha \le \sum_{\beta=1}^{3} \Gamma_\beta \ (\forall \alpha = 1,2,3)$$

d≧3

[Thm*] $$\frac{d}{\sqrt{2}}\Gamma_\alpha \le \sum_{\beta=1}^{d^2-1} \Gamma_\beta \ (\forall \alpha = 1,\dots,d^2-1)$$

[Thm*] $$d\Gamma_\alpha \le \sum_{\beta=1}^{d^2-1} \Gamma_\beta \ (\forall \alpha = 1,\dots,d^2-1)$$

[Thm*] $$\frac{2d}{1+\sqrt{2(1-\frac{1}{d})}}\Gamma_\alpha \le \sum_{\beta=1}^{d^2-1} \Gamma_\beta \ (\forall \alpha = 1,\dots,d^2-1)$$

To ... best constant c(d)

$$c(d)\Gamma_\alpha \le \sum_{\beta=1}^{d^2-1} \Gamma_\beta$$

P. Muratore-Ginanneschi, G. K., D. Chruściński (2024).

D. Chruściński, F. vom Ende, G. K., P. Muratore-Ginanneschi (2025).

* Kimura (2012) * Kimura, Ajisaka, Watabe (2017) * Chruściński, Kimura, Kossakowski, Shishido (2020) * Chruściński, Fujii, Kimura, Ohno (2021)

[Conjecture] (CKKS 2020) For any d-level GKLS,

$$d\Gamma_\alpha \leq \sum_{\beta=1}^{d^2-1} \Gamma_\beta \ (\forall\alpha = 1, \ldots, d^2 - 1)$$

* Numerically supported
* If this is true, c(d)=d is the best bound !!
* Satisfied in important classes of quantum dynamical semigroup

D. Chruscinski, G. K., A. Kossakowski, Y. Shishido (2021)

A Tight Model

$$\mathcal{L}(\rho) = 2L\rho L - L^2\rho - \rho L^2 \quad (L^\dagger = L)$$

$$\mathcal{L}(|k\rangle\langle l|) = -(E_k - E_l)^2 |k\rangle\langle l| \qquad (E_i\text{: Eigenvalues of } L)$$

$$\sum_{\beta=1}^{d^2-1} \Gamma_\beta \geq d\Gamma_\alpha \iff 2\sum_{k<l}(E_k - E_l)^2 \geq d(E_i - E_j)^2$$

The best constant would be d

oreover, the equality is attained iff

$$E_2 = \cdots = E_{d-1} = \frac{E_1 + E_d}{2}$$

To find best constant c(d)

$$c(d)\Gamma_\alpha \leq \sum_{\beta=1}^{d^2-1} \Gamma_\beta \Rightarrow c(d) \leq d$$

Proof based on

[Lemma] For any $x_1, x_2, \ldots, x_n \in [a, b]$,

$$2\sum_{i=1}^{n}((x_i - a)^2 + (x_i - b)^2) + \sum_{i,j=1}^{n}(x_i - x_j)^2 \geq n(a-b)^2,$$

where the equality holds iff $x_1 = \cdots = x_n = \frac{a+b}{2}$.

* Chruściński, Kimura, Kossakowski, Shishido (2021)

Class of Covariant Generator

$$U_{\boldsymbol{x}} \mathcal{L}(X) U_{\boldsymbol{x}}^{\dagger} = \mathcal{L}(U_{\boldsymbol{x}} X U_{\boldsymbol{x}}^{\dagger})$$

where $U_{\boldsymbol{x}} = \sum_{k=1}^{d} e^{-ix_k} |k\rangle\langle k|$ and $\boldsymbol{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d$.

[Theorem] For any covariant GKLS generator,

$$d\Gamma_\alpha \leq \sum_{\beta=1}^{d^2-1} \Gamma_\beta \ (\forall \alpha = 1, \ldots, d^2 - 1)$$

Including
* Pauli Master equation
* Weakly interacting model with non-degenerate invariant state (Davies 1974)

Class of entropy non-decreasing $\frac{d}{dt}S(\rho_t) \geq 0$

$\Leftrightarrow$ Class of unital semigroup $\Lambda_t(\mathbb{I}) = \mathbb{I}$

Benatti (1988), Aniello, Chruscinski (2016)

[Theorem] For any unital generator,

$$d\Gamma_\alpha \leq \sum_{\beta=1}^{d^2-1} \Gamma_\beta \ (\forall \alpha = 1, \ldots, d^2 - 1)$$

[Proof] Omit. But through proving this,
we have found a nice characterization of relaxation rate.

Characterization of Relaxation Rate

* Invariant state $\mathcal{L}(\omega) = 0$ ← Fixed Point Theorem
* Eigen equation $\mathcal{L}(u_\alpha) = \lambda_\alpha u_\alpha$

[Prop.] For any GKLS generator

$$\Gamma_\alpha = \frac{1}{2\|u_\alpha\|_\omega^2} \sum_k \|[L_k, u_\alpha]\|_\omega^2$$

where $\|X\|_\omega^2 := \mathrm{tr}(\omega X^\dagger X)$

[Proof] GKLS rep. reads: $\mathcal{L}^\dagger(X^\dagger X) - \mathcal{L}^\dagger(X^\dagger)X - X^\dagger \mathcal{L}^\dagger(X) = \sum_k [L_k, X]^\dagger [L_k, X]$ $(\mathrm{Tr}(X\mathcal{L}(Y)) = \mathrm{Tr}(\mathcal{L}^\dagger(X)Y))$

Taking $X = u_\alpha$ $\mathcal{L}^\dagger(u_\alpha^\dagger u_\alpha) + 2\Gamma_\alpha u_\alpha^\dagger u_\alpha = \sum_k [L_k, u_\alpha]^\dagger [L_k, u_\alpha]$

which implies $\mathrm{tr}(\omega \mathcal{L}^\dagger(u_\alpha^\dagger u_\alpha)) + 2\Gamma_\alpha \mathrm{tr}(\omega u_\alpha^\dagger u_\alpha) = \sum_k \mathrm{tr}(\omega [L_k, u_\alpha]^\dagger [L_k, u_\alpha]),$

Since $\mathcal{L}(\omega) = 0$, first term vanishes.

Approach based on r-function

[Definition] (r-function) For complex matrices $A, B \in M_d(\mathbb{C})$, we define

$$r(A,B) := \frac{1}{2}\operatorname{tr}(A^\dagger A B^\dagger B + A A^\dagger B^\dagger B - A^\dagger B A B^\dagger - B A^\dagger B^\dagger A)$$

$$\begin{aligned} r(A,B) &= \frac{1}{2}\operatorname{tr}(\{A,A^\dagger\}B^\dagger B) - \Re\operatorname{tr}(A^\dagger B A B^\dagger), \\ &= \frac{1}{2}(\langle [B,A] | BA\rangle + \langle [B,A^\dagger] | BA^\dagger\rangle), \\ &= \frac{1}{2}(\|[A,B]\|^2 + \operatorname{tr} A^\dagger A [B^\dagger, B]) \\ &= \frac{1}{2}(\|[A^\dagger,B^\dagger]\|^2 + \operatorname{tr} A^\dagger A [B^\dagger, B]), \\ &= \frac{1}{2}(\|[A,B^\dagger]\|^2 + \operatorname{tr} A A^\dagger [B^\dagger, B]) \\ &= \frac{1}{2}(\|[A^\dagger,B]\|^2 + \operatorname{tr} A A^\dagger [B^\dagger, B]), \\ &= \frac{1}{4}(\|[A,B]\|^2 + \|[A^\dagger,B]\|^2 + \operatorname{tr}(\{A,A^\dagger\}[B^\dagger,B])). \end{aligned}$$

For normal B, $r(A,B) = \frac{1}{2}\|[A,B]\|^2$

For Cartesian decomposition $A = A_R + iA_I$, $r(A,B) = r(A_R,B) + ir(A_I,B)$

Commutator

$$[A,B] := AB - BA$$

Anti-commutator

$$\{A,B\} := AB + BA$$

Hilbert-Schmidt Inner Prod.

$$\langle A, B\rangle := \operatorname{tr} A^\dagger B$$

Frobenius (Hilbert-Schmidt) Norm

$$\|A\| := \sqrt{\operatorname{tr} A^\dagger A}$$

D. Chruscinski, R. Fujii, G. K., H. Ohno (2021)

Approach based on r-function

$$\sum_{\alpha=1}^{d^2-1} \Gamma_\alpha = d \sum_k \|L_k\|^2$$

[Prop.] $$\Gamma_\alpha = \frac{1}{\|u_\alpha\|^2} \sum_k r(u_\alpha, L_k)$$

$$\mathcal{L} u_\alpha = \lambda_\alpha u_\alpha \ (u_\alpha \neq 0)$$

[Proof] $\Gamma_\alpha := -\mathrm{Re}\lambda_\alpha$ & $\|u_\alpha\|^2 = \mathrm{tr} u_\alpha^\dagger u_\alpha$

$$\mathrm{tr} \left(u_\alpha^\dagger \times \lambda_\alpha u_\alpha = \mathcal{L}(u_\alpha) = -i[H, u_\alpha] + \frac{1}{2} \sum_k (2 L_k u_\alpha L_k^\dagger - L_k^\dagger L_k u_\alpha - u_\alpha L_k^\dagger L_k) \right)$$

- Re

$$\Gamma_\alpha = \frac{1}{2\|u_\alpha\|^2} \sum_k \mathrm{tr}(u_\alpha^\dagger u_\alpha L_k^\dagger L_k + u_\alpha u_\alpha^\dagger L_k^\dagger L_k - u_\alpha^\dagger L_k u_\alpha L_k^\dagger - L_k u_\alpha^\dagger L_k^\dagger u_\alpha)$$

$$= \frac{1}{\|u_\alpha\|^2} \sum_k r(u_\alpha, L_k)$$

* Complex Eigenvalues appear as conjugate pair

Hermitian Preserving Property

$$\Gamma_\alpha := -\mathrm{Re}\lambda_\alpha$$

$$\sum_{\alpha=1}^{d^2-1} \Gamma_\alpha = -\sum_{\alpha=0}^{d^2-1} \lambda_\alpha = -\mathrm{tr}\mathcal{L}$$

* Trace of generator

$$\mathrm{tr}\mathcal{L} = \sum_k (|\mathrm{tr}L_k|^2 - d\|L_k\|^2)$$

Frobenius Norm:

$$\|A\| := \sqrt{\mathrm{tr}\, A^\dagger A}$$

Without loss of generality

$$\mathrm{tr}\, L_k = 0$$

$$\mathrm{tr}\mathcal{L} = -d\sum_k \|L_k\|^2$$

Wolf and Cirac (2008)

Kimura Ajisaka Watanabe (2017)

[Prop]

$$\sum_{\alpha=1}^{d^2-1} \Gamma_\alpha = d\sum_k \|L_k\|^2$$

Approach based on r-function

[Prop.] $$\Gamma_\alpha = \frac{1}{\|u_\alpha\|^2}\sum_k r(u_\alpha, L_k)$$

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$$\mathrm{tr}\left(u_\alpha^\dagger \times \ \lambda_\alpha u_\alpha = \mathcal{L}(u_\alpha) = -i[H, u_\alpha] + \frac{1}{2}\sum_k (2L_k u_\alpha L_k^\dagger - L_k^\dagger L_k u_\alpha - u_\alpha L_k^\dagger L_k) \right)$$

- Re

$$\Gamma_\alpha = \frac{1}{2\|u_\alpha\|^2}\sum_k \mathrm{tr}(u_\alpha^\dagger u_\alpha L_k^\dagger L_k + u_\alpha u_\alpha^\dagger L_k^\dagger L_k - u_\alpha^\dagger L_k u_\alpha L_k^\dagger - L_k u_\alpha^\dagger L_k^\dagger u_\alpha)$$

$$= \frac{1}{\|u_\alpha\|^2}\sum_k r(u_\alpha, L_k)$$

$$\sum_{\alpha=1}^{d^2-1}\Gamma_\alpha = d\sum_k \|L_k\|^2$$

$$r(A, B) \leq c(d)\|A\|^2\|B\|^2$$

$$\Gamma_\alpha \leq \frac{1}{\|u_\alpha\|^2}\sum_k c(d)\|u_\alpha\|^2\|L_k\|^2$$

$$= c(d)\sum_k \|L_k\|^2 = \frac{c(d)}{d}\sum_\alpha \Gamma_\alpha$$

Universal Constraint !!

$$\frac{d}{c(d)}\Gamma_\alpha \leq \sum_\beta \Gamma_\beta$$

[Prop] For any matrices A, B,

$r($

$$(\Delta A)^2(\Delta B)^2 \geq \frac{1}{4}\left|\langle [A,B]\rangle\right|^2 + \mathrm{Cov}(A,B)^2 + \frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2}\langle \left|[A,B]\right|^2\rangle$$

Proof.

$$r(A,B) = \frac{1}{2}(\langle [B,A]|BA\rangle + \langle [$$

$$\leq \frac{1}{2}(\|[B,A]\|\|B\|\|A\| + \|[B,A^\dagger]\|\|B\|\|A^\dagger\|$$

(using triangle, Schwarz, and norm inequalities.,)

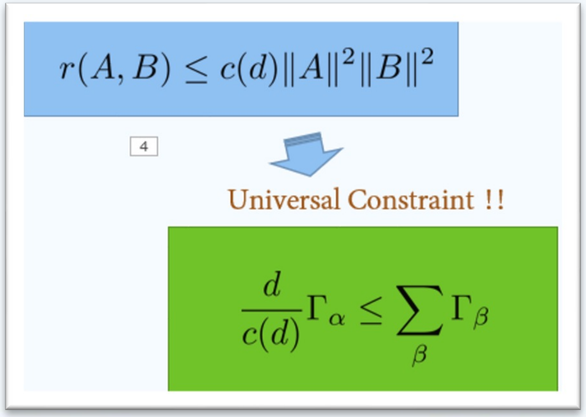
$$\leq \sqrt{2}\|A\|^2\|B\|^2$$

$$\|[A,B]\|^2 \leq 2\|A\|^2\|B\|^2$$

(Böttcher-Wenzel Inequality)

Series of new uncertainty relations

arXiv:2504.20404
arXiv:2505.19861
arXiv:2406.12280
arXiv:2403.04199



[Theorem] (KAW) For any d-level GKLS,

$$\frac{d}{\sqrt{2}}\Gamma_\alpha \leq \sum_{\beta=1}^{d^2-1}\Gamma_\beta$$

Kimura, Ajisaka, Watabe (2017)

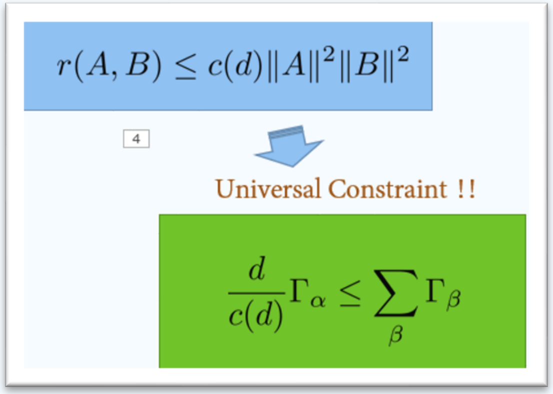
Approach based on r-function

[Prop.] For any complex matrices $A, B \in M_d(\mathbb{C})$,

$$r(A,B) \leq \frac{1+\sqrt{2}}{2}\|A\|^2\|B\|^2,$$

where the equality can be achieved by a self-adjoint A.

[Proof] Omit. But we have first show this where A is normal first, and then used relation $r(A,B) = r(A_R,B)+i\, r(A_I,B)$



[Theorem] (CFKO[1]) For any d-level GKLS,

$$\frac{\sqrt{2}d}{1+\sqrt{2}}\Gamma_\alpha \leq \sum_{\beta=1}^{d^2-1}\Gamma_\beta \ (\forall \alpha = 1,\ldots,d^2-1)$$

D. Chruscinski, R. Fujii, G. K., H. Ohno (2021).

Approach based on r-function

Trace Preserving Property

**Eigenvector belonging to non-zero eigenvalue is traceless $\lambda_\alpha \neq 0 \Rightarrow \mathrm{tr}\ u_\alpha = 0$

[Prop.] For any complex matrices $A, B \in M_d(\mathbb{C})$ with $\mathrm{tr}A = 0$,

$$r(A,B) \leq \frac{1+\sqrt{2(1-\frac{1}{d})}}{2}\|A\|^2\|B\|^2$$

where the equality can be achieved by a self-adjoint A.

$r(A,B) \leq c(d)\|A\|^2\|B\|^2$

4

Universal Constraint !!

$$\frac{d}{c(d)}\Gamma_\alpha \leq \sum_\beta \Gamma_\beta$$

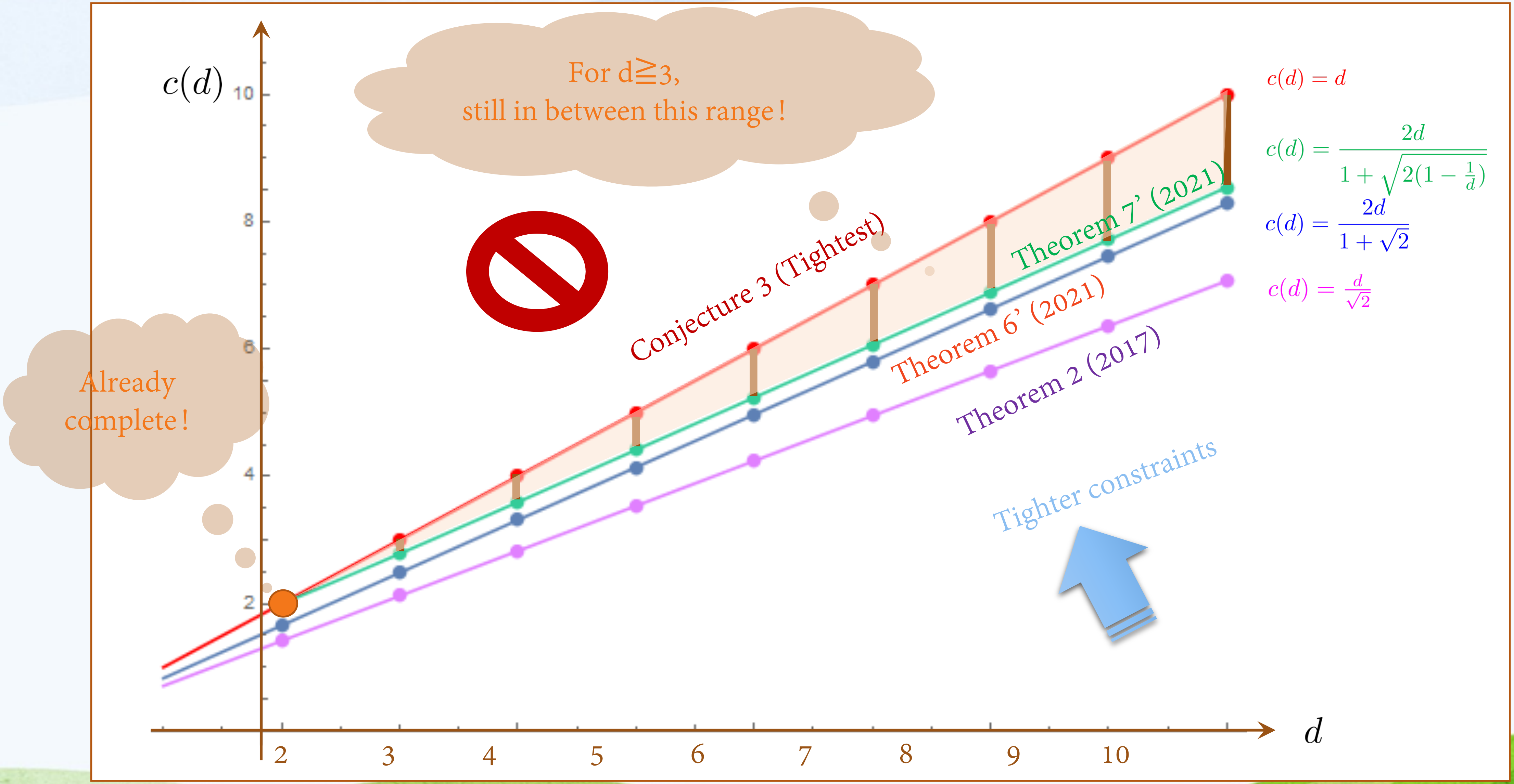
[Theorem] (CFKO²) For any d-level GKLS,

$$\frac{2d}{1+\sqrt{2(1-\frac{1}{d})}}\Gamma_\alpha \leq \sum_{\beta=1}^{d^2-1}\Gamma_\beta \ (\forall \alpha = 1,\ldots,d^2-1)$$

D. Chruscinski, R. Fujii, G. K., H. Ohno (2021).

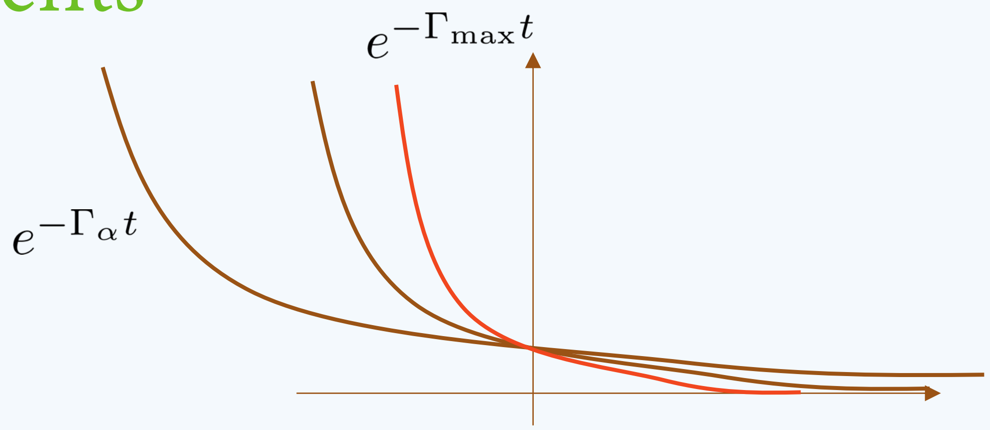
Universal Constraints for GKLS generator

$$c(d)\Gamma_\alpha \leq \sum_{\beta=1}^{d^2-1} \Gamma_\beta$$



Approach based on Lyapunov exponents

$$\boldsymbol{\rho}(t) = \sum_{i=1}^{d} p_i(t) |\boldsymbol{\psi}_i(t)\rangle\langle\boldsymbol{\psi}_i(t)|$$



For sufficiently past time t ≃ -∞

$$\lim_{t\to-\infty} p_i(t) = e^{-\Gamma_{\max} t} q_i(t) + \cdots$$

Lyapunov exponents

$$\chi := \lim_{t\downarrow-\infty} \sup_t \left(-\frac{1}{t} \ln \|\boldsymbol{p}(t)\| \right) = \Gamma_{\max}$$

Ginanneschi, Kimura,Chruściński (2024)

Approach based on Lyapunov exponents

Teich and Mahler (1992) $\frac{dp_i}{dt} = \sum_j W_{ij}(t) p_j(t)$

where $W_{ij}(t) = R_{ij}(t) - \delta_{ij} \sum_{k=1}^{d} R_{kj}(t)$ and $R_{ij}(t) = \sum_{n=1}^{d^2-1} \gamma_n \, |\langle \boldsymbol{\psi}_i(t) \, L_n \boldsymbol{\psi}_j(t) \rangle|^2 .$

$$\mathcal{D}(\rho) = \frac{1}{2} \sum_k \gamma_k (2 L_k \rho L_k^\dagger - L_k^\dagger L_k \rho - \rho L_k^\dagger L_k)$$

e.g., Adrianova (1995) $\Gamma_{\max} = \chi \leq \sup_{t \in \mathbb{R}} \|W(t)\| \leq c(d) \sum_\beta \Gamma_\beta$

$$\|W(t)\| \leq c(d) \sum_\beta \Gamma_\beta$$

Different Norms gives Different Bounds !

$\|A\|_1 := \max_j \sum_i |a_{ij}|$ (Column-sum Norm) → $\|W(t)\|_1 \leq \frac{2}{d}\sum_\beta \Gamma_\beta$ (Wolf Cirac 2008)

$\|A\|_2 := \sqrt{\sum_{i,j} |a_{ij}|^2}$ (Frobenius Norm)

$\|A\|_\infty := \max_i \sum_j |a_{ij}|$ (Row-sum Norm) → $\|W(t)\|_\infty \leq \frac{1}{d}\sum_\beta \Gamma_\beta$ (Tight Bound !!)

[Theorem] (GKC) For any d-level GKLS, the tight constraint is given by

$$d\Gamma_\alpha \leq \sum_{\beta=1}^{d^2-1} \Gamma_\beta \ (\forall \alpha = 1, \ldots, d^2-1)$$

Ginanneschi, Kimura,Chruściński (2024)

But according to discipline of natural science,

we must rely on **experiments** in order to determine something is right or wrong !!

Experiments

What is the **physics of CP** condition ??

Bell Theorem

Reality

Locality

"Free will"

Bell's Inequality

$$\langle AC\rangle + \langle AD\rangle + \langle BC\rangle - \langle BD\rangle \leq 2$$

Goal

CP

Markovianity

Experimentally testable condition.

$$d\Gamma_\alpha \leq \sum_{\beta=1}^{d^2-1} \Gamma_\beta$$

What Happens Beyond CP ?

Ancilla Witness ?

Surprisingly…
the same constraints
for all n ≧ 2

Constraints on Relaxation Times
for Markovian n-Positive Quantum Dynamics

CPTP map as a Time evolution map

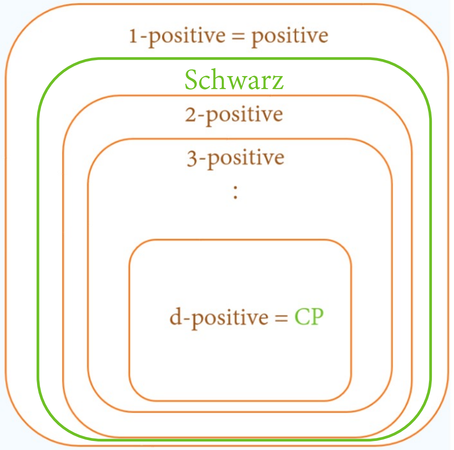
Initial State $\rho \mapsto \Phi(\rho)$ Final State

[Def] Φ is n-positive (n =1,2,3,…,) if

$\Phi \otimes \mathrm{Id}_n$ is positive on $\mathcal{H} \otimes \mathbb{C}^n$ system $\mathcal{H} = \mathbb{C}^d$ $\otimes$ $\mathbb{C}^n$

[Def] Φ is completely positive if Φ is n-positive **for all n =1,2,3,…**

[Remarks] (i) 1-positive ⇔ positive
(ii) d-positive ⇔ CP
(iii) for all k =1,…,d-1, there are k-positive but not (k+1)-positive



$$c(d)\Gamma_\alpha \leq \sum_{\beta=1}^{d^2-1} \Gamma_\beta$$

k-positive	c(d)
1 = positive	1
3/2 = Schwarz	(d+1)/2
2	d
3	d
:	:
d-1	d
d = CP	d

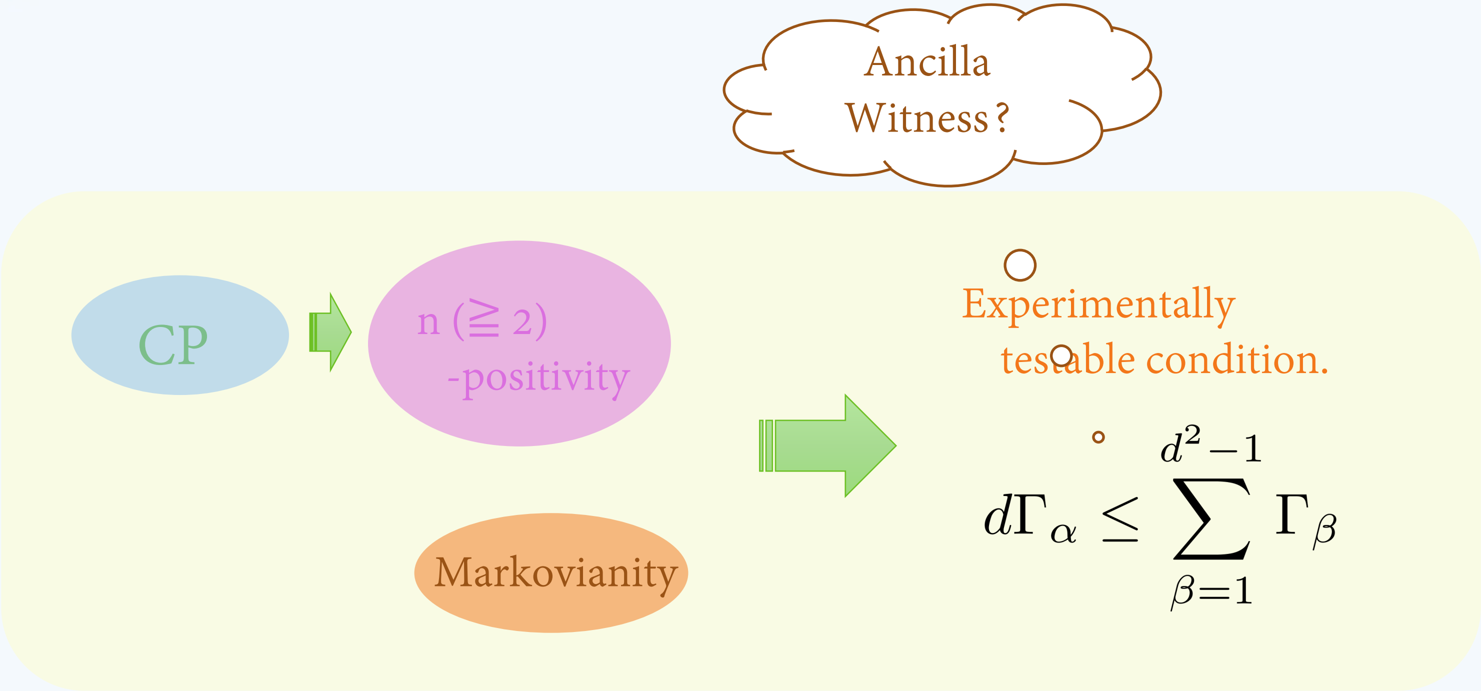
[Def] Unital linear map Ψ is called Schwarz map iff $\Psi(X^\dagger X) \geq \Psi(X)^\dagger \Psi(X) \quad \forall X \in \mathcal{L}(\mathcal{H})$

[Def] A dynamical semigroup $\{\Phi_t\}_{t\geq 0}$ is called of Schwarz class iff $\Phi_t^\dagger$ is a Schwarz map.

D. Chruscinski, G. K., F. Mukhamedov (2024).
D. Chruściński, F. vom Ende, G. K., P. Muratore-Ginanneschi (2025).

But according to discipline of natural science,

we must rely on **experiments** in order to determine something is right or wrong !!



If these constraints were violated experimentally,
one could conclude that the observation cannot be explained not only by a **CP dynamical semigroup** (i.e., a GKLS dynamics)
but also by any **n-positive dynamical semigroup** with $n \geq 2$.

Sketch of Proof.

$\mathcal{L}$: a generator of n ($\geqq$ 2)-positive dynamical semigroup

W.L.G. by continuity argument

Step 1: $\langle \mathfrak{L}^{\#}(A), B\rangle_{\omega} := \langle A, \mathfrak{L}^{\dagger}(B)\rangle_{\omega}$

ω : a faithful invariant state $\mathcal{L}(\omega) = 0$

Kubo-Martin-Schwinger inner product $\langle A, B\rangle_{\omega} := \langle A, \omega^{\frac{1}{2}} B \omega^{\frac{1}{2}}\rangle$

[Prop] $\widetilde{\mathfrak{L}} := \frac{1}{2}(\mathfrak{L}^{\dagger} + \mathfrak{L}^{\#})$ is a self-adjoint generator of n-positive dynamical semigroup

[Bendixson--Hirsch inequality]

For any matrix A with (complex) eigenvalues $\{a_1, \ldots, a_d\}$

$$m \leq \mathrm{Re}\, a_k \leq M$$

where m and M are minimal and maximal eigenvalues of the Hermitian matrix

$$\frac{1}{2}(A + A^{\dagger})$$

[Corollary] It is enough to show $d\Gamma_{\alpha} \leq \sum_{\beta=1}^{d^2-1} \Gamma_{\beta}$ for a self-adjoint generator

In particular, one can assume eigenvalues of L are real.

Step 2: For any ONB $\{|e_1\rangle, \dots, |e_d\rangle\}$ in $\mathbb{C}^d$, $\mathcal{K}_{ij} := \mathrm{Tr}(|e_i\rangle\langle e_i|\mathfrak{L}(|e_j\rangle\langle e_j|))$

[Prop ♧] $\mathcal{K}$ is a classical generator, hence real parts of eigenvalues are non-positive.

[Prop ♤] A real eigenvalue λ of $\mathcal{L}$ is an eigenvalue of $\mathcal{K}$ with some ONB.

∵ (i) One finds Hermitian eigenmatrix $X = \sum_i x_i |x_i\rangle\langle x_i|$ for real eigenvalue λ

(ii) $$(\mathcal{K}\boldsymbol{x})_i = \sum_{j=1}^{d} \mathcal{K}_{ij} x_j = \sum_{j=1}^{d} \langle x_i|\mathcal{L}(x_j|x_j\rangle\langle x_j|)|x_i\rangle = \langle x_i|\mathcal{L}(X)|x_i\rangle = \lambda\langle x_i|X|x_i\rangle = \lambda x_i.$$

[Prop ♥] $\mathrm{Tr}\,\mathcal{L} \le d\ \mathrm{Tr}\,\mathcal{K}$

$$\Gamma_\alpha = -\lambda_\alpha \overset{♧\&♤}{\le} -\mathrm{Tr}\mathcal{K} \overset{♥}{\le} -\frac{1}{d}\mathrm{Tr}\mathcal{L} = \frac{1}{d}\sum_\beta \Gamma_\beta$$

Proof of ❤️ $\mathrm{Tr}\,\mathcal{L} \leq d\ \mathrm{Tr}\,\mathcal{K}$

[Thm] (Evans, Hanche-Olsen 1979)

$\mathcal{L}$ is a generator of n-positive dynamical semigroup iff it is n-conditional positive:

For any $|\psi\rangle \perp |\phi\rangle \in \mathbb{C}^d \otimes \mathbb{C}^n$, $\langle\psi|\mathcal{L} \otimes \mathrm{Id}(|\psi\rangle\langle\psi|)|\psi\rangle \geq 0$

For mutually orthogonal vectors

$$|\phi_{ij}^{\pm}\rangle = |e_i\rangle \otimes |1\rangle \pm |e_j\rangle \otimes |2\rangle \in \mathbb{C}^d \otimes \mathbb{C}^2$$

$$\sum_{i \neq j} \langle\phi_{ij}^{-}|[\mathcal{L} \otimes \mathrm{Id}](|\phi_{ij}^{+}\rangle\langle\phi_{ij}^{+}|)|\phi_{ij}^{-}\rangle = 2(d\mathrm{Tr}\mathcal{K} - \mathrm{Tr}\mathcal{L}) \geq 0$$

Conclusions and Discussion

Thank you
for your kind attention !!

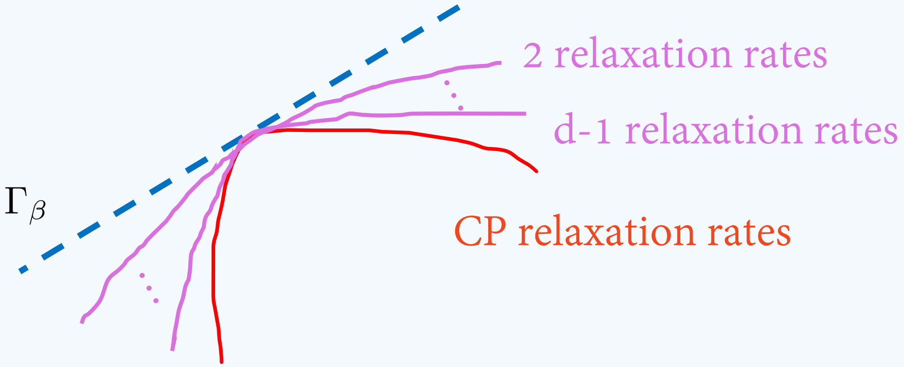
(I) Completed our long-standing problem on the universal constraint on relaxation rates for CP dynamical semigroup !

(II) The constraint is satisfied not only by CP but also n ≧ 2 positive dynamical semigroup,
while Schwarz class gives weaker constraint.

$$c(d)\Gamma_\alpha \leq \sum_{\beta=1}^{d^2-1} \Gamma_\beta$$

k-positive	c(d)
1 = positive	1
3/2 = Schwarz	(d+1)/2
2	d
3	d
:	:
d-1	d
d = CP	d

$$d\Gamma_\alpha \leq \sum_{\beta=1}^{d^2-1} \Gamma_\beta$$



Future Problems !!

* Find (non-linear) constraints on relaxation rates that can distinguish the difference between n and CP cases !!
* Specify the region of Relaxation rates

Another Application 1: Quantum Channel Spectrum

[Thm] Eigenvalues z_α $(\alpha = 1, \ldots, d^2 - 1)$ of any TPCP map satisfies

$$\sum_{\beta=1}^{d^2-1} \mathrm{Re} z_\beta \leq d^2 - 1 + c(d)(\mathrm{Re} z_\alpha - 1)$$

except for $z_0 = 1$

[Proof] If Λ is TPCP, then $\mathcal{L}(\rho) = \Lambda(\rho) - \rho$ is GKLS generator.

Applying $c(d)\Gamma_\alpha \leq \sum_{\beta=1}^{d^2-1} \Gamma_\beta$, we get the inequality.

* For d=2, we have c(2)=2, and this recovers Fujiwara-Algoet condition (1999)

* Chruściński, Kimura, Kossakowski, Shishido (2020)

Another Application 2: Quantum Channel Spectrum

[Fact] Any unitary map is realizable by Schrödinger equation:

$$\mathcal{U}\rho = U\rho U^\dagger \Rightarrow \exists t \ \& \ \mathcal{L}\rho = -i[H,\rho] \text{ s.t. } \mathcal{U} = e^{t\mathcal{L}}$$

[Question] Any CPTP map is realizable by GKLS equation??

NO ! Wolf and Cirac (2008)

* Chruściński, Kimura, Kossakowski, Shishido (2020)

Another Application 2: Quantum Channel Spectrum

[Prop.] A necessary condition for a CPTP map Λ be realizable by GKLS eq. is

$$\det\Lambda \le |u_\alpha|^{c(d)}$$

where u_α is eigenvalue of Λ

[Proof] Obseve that $\sum_{\beta=1}^{d^2-1}\Gamma_\beta \ge c(d)\Gamma_\alpha \Leftrightarrow \exp(-\sum_{\beta=1}^{d^2-1}\Gamma_\beta) \le \exp(-c(d)\Gamma_\alpha) \Leftrightarrow \prod_{\beta=1}^{d^2-1}\exp(-\Gamma_\beta) \le \exp(-c(d)\Gamma_\alpha) \Leftrightarrow \prod_{\beta=1}^{d^2-1}\exp(x_\alpha) \le (\exp(-\Gamma_\alpha))^{c(d)} \Leftrightarrow \prod_{\beta=1}^{d^2-1} u_\beta \le |u_\alpha|^{c(d)} \Leftrightarrow \det\Lambda \le |u_\alpha|^{c(d)}$

* Chruściński, Kimura, Kossakowski, Shishido (2020)

Another Application 2: Quantum Channel Spectrum

For d = 2, any CPTP map Λ is unitary equivalent to

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 & 0 \\ v_1 & \lambda_1 & 0 & 0 \\ v_2 & 0 & \lambda_2 & 0 \\ v_3 & 0 & 0 & \lambda_3 \end{pmatrix}$$

for $1 \geq \lambda_1 \geq \lambda_2 \geq |\lambda_3|$ & $\lambda_1 + \lambda_2 \leq 1 + \lambda_3$

Also sufficient for unital CP map

Fujiwara, Algoet (1999), King, Ruskai (2001)

[Prop.] If a CPTP map Λ is realized by GKLS eq., then

$$\lambda_1 \lambda_2 \lambda_3 \leq |\lambda_3|^2$$

This generalizes Puchała-Rudnicki-Zyczkowski bound (2019)

* Chruściński, Kimura, Kossakowski, Shishido (2020)

Takeaway Message

1) Complete Positivity forces that

any single relaxation rate cannot be too large !!

Complete Positivity

"Markovianity"

Experimentally testable condition.

$$c(d)\Gamma_\alpha \leq \sum_{\beta=1}^{d^2-1} \Gamma_\beta \ (\forall \alpha = 1, \ldots, d^2-1)$$

This may serve as Completely Positive (at least QDS) witness !

Best bound is conjectured to be $c(d)=d$, but still open.

Some Observations

$$\mathcal{L}(u_\alpha) = \lambda_\alpha u_\alpha \ (u_\alpha \neq 0)$$

(1) There is (at least) one zero eigenvalue: $\lambda_0 = 0$

Trace Preserving Property

(2) Complex Eigenvalues appear as conjugate pair

Hermitian Preserving Property

$$\Gamma_\alpha := -\mathrm{Re}\lambda_\alpha$$

$$\sum_{\alpha=1}^{d^2-1} \Gamma_\alpha = -\sum_{\alpha=0}^{d^2-1} \lambda_\alpha = -\mathrm{tr}\mathcal{L}$$

(3) Eigenvector belonging to non-zero eigenvalue is traceless

$$\lambda_\alpha \neq 0 \Rightarrow \mathrm{tr}\ u_\alpha = 0$$

Trace Preserving Property

Some Observations

(4) Trace of generator

$$\mathrm{tr}\mathcal{L} = \sum_k (|\mathrm{tr} L_k|^2 - d\|L_k\|^2)$$

Frobenius Norm:

$$\|A\| := \sqrt{\mathrm{tr}\, A^\dagger A}$$

Without loss of generality

$$\mathrm{tr}\, L_k = 0$$

$$\mathrm{tr}\mathcal{L} = -d \sum_k \|L_k\|^2$$

Wolf and Cirac (2008)

Kimura Ajisaka Watanabe (2017)

Tensor rep. by $|E_{ij}\rangle\!\rangle\langle\!\langle E_{kl}| \hat{\mapsto} E_{ik} \otimes E_{jl}$ where $E_{ij} = |i\rangle\langle j|$

$$\mathcal{L}_A X := AX \mapsto \hat{\mathcal{L}}_A = A \otimes \mathbb{I} \quad \mathcal{R}_A X := XA \mapsto \hat{\mathcal{R}}_A = \mathbb{I} \otimes A^T$$

$$\mathcal{L}(\rho) = -i[H,\rho] + \frac{1}{2}\sum_k (2L_k \rho L_k^\dagger - L_k^\dagger L_k \rho - \rho L_k^\dagger L_k)$$

$$\mapsto \hat{\mathcal{L}} = -i(I \otimes H - H^T \otimes I) + \frac{1}{2}\sum_k (2\overline{L_k} \otimes L_k - \mathbb{I} \otimes L_k^\dagger L_k - L_k^T \overline{L_k} \otimes \mathbb{I})$$

Some Observations

Combining

$$\sum_{\alpha=1}^{d^2-1} \Gamma_\alpha = -\sum_{\alpha=0}^{d^2-1} \lambda_\alpha = -\mathrm{tr}\mathcal{L}$$

+

$$\mathrm{tr}\mathcal{L} = -d\sum_k ||L_k||^2$$

[Prop]

$$\sum_{\alpha=1}^{d^2-1} \Gamma_\alpha = d\sum_k \|L_k\|^2$$